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ADVANCED RISK AND PORTFOLIO MANAGEMENT BOOTCAMP

PREREQUISITES CHECK

Exercise 1 *Transformation of a random variable*

Consider as in $(T1.1)$ in the Technical Appendices at `www.symmys.com/node/198` the following transformation of the generic random variable X:

$$X \mapsto Y \equiv g(X), \tag{1}$$

where g is an increasing and thus invertible function.

Prove the following formulas:

$$F_Y(y) = F_X\left(g^{-1}(y)\right) \tag{2}$$

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}. \tag{3}$$

$$Q_Y(p) = g\left(Q_X(p)\right) \tag{4}$$

See Section 1.1 in the Technical Appendices. In particular, for the cdf, by the definition of the cumulative distribution function F_Y we have:

$$\begin{aligned} F_Y(y) &\equiv \mathbb{P}\{Y \leq y\} = \mathbb{P}\{g(X) \leq y\} \\ &= \mathbb{P}\left\{X \leq g^{-1}(y)\right\} \\ &= F_X\left(g^{-1}(y)\right). \end{aligned} \tag{5}$$

For the pdf, by derivation of both sides the above result we obtain:

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \tag{6}$$

For the quantile, as in the Technical Appendices consider the following series of identities:

$$F_Y\left(g\left(Q_X(p)\right)\right) = \mathbb{P}\left\{Y \leq g\left(Q_X(p)\right)\right\} = \mathbb{P}\left\{X \leq Q_X(p)\right\} = p, \tag{7}$$

By applying the definition of the quantile Q_Y to the above terms we obtain:

$$Q_Y(p) = g\left(Q_X(p)\right), \tag{8}$$

Exercise 2 *Affine transformation of a random variable*

Consider as in ($T1.12$) in the Technical Appendices the following positive affine transformation of the generic random variable X:

$$X \mapsto Y \equiv g(X) \equiv m + sX, \tag{9}$$

where $s > 0$ and m is a generic constant.

Prove the following formulas:

$$f_Y(y) = \frac{1}{s} f_X\left(\frac{y-m}{s}\right) \tag{10}$$

$$F_Y(y) = F_X\left(\frac{y-m}{s}\right) \tag{11}$$

$$Q_Y(p) = m + sQ_X(p) \tag{12}$$

$$\phi_Y(\omega) = e^{i\omega m}\phi_X(s\omega) \tag{13}$$

See Section 1.2 in the Technical Appendices.

Exercise 3 *Normal distribution*

Consider as in (1.66) in the textbook a normal random variable

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right). \tag{14}$$

Compute μ and σ^2 such that $\mathrm{E}\{X\} \equiv 3$ and $\mathrm{Var}\{X\} \equiv 5$.

From (1.71) in the textbook, $\mathrm{E}\{X\} = \mu$ and from (1.72) in the textbook, $\mathrm{Var}\{X\} = \sigma^2$.

Exercise 4 *Lognormal distribution*

Consider as in (1.94) in the textbook a lognormal random variable

$$X \sim \mathrm{LogN}\left(\mu, \sigma^2\right). \tag{15}$$

Compute μ and σ^2 such that $\mathrm{E}\{X\} \equiv 3$ and $\mathrm{Var}\{X\} \equiv 5$.

From (1.98) and (1.99) in the textbook we need to solve for μ and σ^2 the following system:

$$E = e^{\mu + \frac{\sigma^2}{2}} \tag{16}$$

$$V = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) \tag{17}$$

or

$$2\ln(E) = 2\mu + \sigma^2 \tag{18}$$

$$\ln(V) = 2\mu + \sigma^2 + \ln\left(e^{\sigma^2} - 1\right) \tag{19}$$

Therefore

$$\ln\left(\frac{V}{E^2}\right) = \ln\left(e^{\sigma^2} - 1\right) \tag{20}$$

or

$$\sigma^2 = \ln\left(1 + \frac{V}{E^2}\right). \tag{21}$$

From (18) we then obtain:

$$\mu = \ln(E) - \frac{1}{2}\ln\left(1 + \frac{V}{E^2}\right). \tag{22}$$

Exercise 5 *Student t distribution*

Consider as in (1.85) in the textbook a Student t random variable

$$X \sim \text{St}\left(\nu, \mu, \sigma^2\right). \tag{23}$$

Knowing that $\nu \equiv 4$, compute μ and σ^2 such that $\text{E}\{X\} \equiv 3$ and $\text{Var}\{X\} \equiv 5$.

From (1.89) in the textbook $\text{E}\{X\} = \mu$ and from (1.90)

$$\sigma^2 = \text{Var}\{X\}\frac{\nu - 2}{\nu} \tag{24}$$

Exercise 6 *Gamma vs. chi-square distributions*

Consider as in (1.107) in the textbook a gamma-distributed random variable

$$X \sim \text{Ga}\left(\nu, \mu, \sigma^2\right). \tag{25}$$

We recall that such variable is defined in distribution as follows

$$X \stackrel{d}{\equiv} Y_1^2 + \cdots + Y_\nu^2, \tag{26}$$

where

$$Y_1 \stackrel{d}{\equiv} \cdots \stackrel{d}{\equiv} Y_\nu \sim \text{N}\left(\mu, \sigma^2\right) \tag{27}$$

are independent. For which values of ν, μ and σ^2 does this distribution coincide with the chi-square distribution with ten degrees of freedom?

For $\nu \equiv 10$, $\mu \equiv 0$ and $\sigma^2 \equiv 1$ we obtain $X \sim \chi^2_{10}$, see (1.109) in the textbook.

Exercise 7 *Characteristic function*

Consider the random variable defined in distribution as

$$X \overset{d}{\equiv} Y + Z, \tag{28}$$

where Y and Z are independent.

Compute the characteristic function ϕ_X of X from the characteristic functions ϕ_Y of Y and ϕ_Z of Z.

$$\begin{aligned} \phi_X(\omega) &\equiv \mathrm{E}\left\{e^{i\omega X}\right\} = \mathrm{E}\left\{e^{i\omega(Y+Z)}\right\} = \mathrm{E}\left\{e^{i\omega Y} e^{i\omega Z}\right\} \\ &= \mathrm{E}\left\{e^{i\omega Y}\right\} \mathrm{E}\left\{e^{i\omega Z}\right\} = \phi_Y(\omega)\, \phi_Z(\omega) \end{aligned} \tag{29}$$

Exercise 8 *Statistical testing*

Consider a time series of independent and identically distributed random variables

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right), \quad t = 1, \ldots, T. \tag{30}$$

Consider the sample mean

$$\widehat{\mu} \equiv \frac{1}{T} \sum_{t=1}^{T} X_t. \tag{31}$$

Compute the distribution of $\widehat{\mu}$.

We can write

$$\begin{pmatrix} X_1 \\ \vdots \\ X_T \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & \sigma^2 \end{pmatrix}\right). \tag{32}$$

From (2.163) in the textbook, which we report here

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{a} + \mathbf{B}\mathbf{X} \sim \mathrm{N}\left(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\right)$$

for any conformable vector and matrix $\mathbf{a}$ and $\mathbf{B}$ respectively, it follows

$$\widehat{\mu} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{T}\right). \tag{33}$$

What is the probability that the sample mean (31) exceed a given value $\widetilde{\mu}$?

$$\begin{aligned}\mathbb{P}\{\widehat{\mu} > \widetilde{\mu}\} &= 1 - \mathbb{P}\{\widehat{\mu} \leq \widetilde{\mu}\} \\ &= 1 - \mathbb{P}\left\{\frac{\widehat{\mu} - \mu}{\sqrt{\sigma^2/T}} \leq \frac{\widetilde{\mu} - \mu}{\sqrt{\sigma^2/T}}\right\}\end{aligned} \tag{34}$$

From () and (33) it follows

$$\frac{\widehat{\mu} - \mu}{\sqrt{\sigma^2/T}} \sim \mathrm{N}(0,1), \tag{35}$$

therefore

$$\mathbb{P}\{\widehat{\mu} > \widetilde{\mu}\} = 1 - \Phi\left(\frac{\widetilde{\mu} - \mu}{\sqrt{\sigma^2/T}}\right), \tag{36}$$

where Φ denotes the cdf of the standard normal distribution.

Exercise 9 *Matrix algebra*

Consider a $N \times N$ matrix of the form

$$\mathbf{\Sigma} \equiv \mathbf{E}\mathbf{\Lambda}\mathbf{E}', \tag{37}$$

where $\mathbf{\Lambda}$ is diagonal and $\mathbf{E}$ is invertible.

Prove that $\mathbf{\Sigma}$ is symmetric, see definition $(A.51)$ in the textbook, available at `www.symmys.com/node/89`

$$\mathbf{\Sigma}' \equiv \left(\mathbf{E}\mathbf{\Lambda}\mathbf{E}'\right)' = \mathbf{E}\mathbf{\Lambda}'\mathbf{E}' = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' = \mathbf{\Sigma} \tag{38}$$

Prove that $\mathbf{\Sigma}$ is positive if and only if all the diagonal elements of $\mathbf{\Lambda}$ are positive, see definition $(A.52)$ in the textbook.

For any $\mathbf{v}$ there exists one and only one $\mathbf{w} \equiv \mathbf{E}'\mathbf{v}$ and $\mathbf{w} \equiv \mathbf{0} \iff \mathbf{v} \equiv \mathbf{0}$. Assume that all the diagonal elements of $\mathbf{\Lambda}$ are positive and $\mathbf{v} \neq \mathbf{0}$. Then:

$$\begin{aligned}\mathbf{v}'\mathbf{\Sigma}\mathbf{v} &\equiv \mathbf{v}'\mathbf{E}\mathbf{\Lambda}\mathbf{E}'\mathbf{v} = \mathbf{w}'\mathbf{\Lambda}\mathbf{w} \\ &= \sum_{n=1}^{N} w_n^2 \lambda_n > 0.\end{aligned} \tag{39}$$

Similarly, from the above identities, if $0 < \mathbf{v}'\mathbf{\Sigma}\mathbf{v}$ for any $\mathbf{v} \neq \mathbf{0}$, then each λ_n has to be positive.