Attilio Meucci

www.symmys.com

Handling Estimation Risk in Portfolio Optimization

AGENDA

ESTIMATION RISK

ROBUST OPTIMIZATION

SHRINKAGE OPTIMIZATION

BAYESIAN OPTIMIZATION

REFERENCES

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ESTIMATION RISK – portfolio optimization: theory …

$$\boldsymbol{w}_v \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \{\boldsymbol{w}'\boldsymbol{m}\}$$

subject to $\begin{cases} \boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \leq v \\ \boldsymbol{w} \in \mathcal{C} \end{cases}$

$\boldsymbol{m} \equiv \mathrm{E}\{\boldsymbol{R}_{T+\tau}\}$: **expected returns**

$\boldsymbol{S} \equiv \mathrm{Cov}\{\boldsymbol{R}_{T+\tau}\}$: **covariance**

$\boldsymbol{w}$: **relative portfolio weights**

v : **significant grid of target variances**

$\mathcal{C}$: **investment constraints, e.g.** $\boldsymbol{w}'\boldsymbol{1} = 1,\ \boldsymbol{w} \geq \boldsymbol{0}$

ESTIMATION RISK – portfolio optimization: … practice

$$\boldsymbol{w}_v \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}' \boldsymbol{m} \right\}$$

subject to $\begin{cases} \boldsymbol{w}' \boldsymbol{S} \boldsymbol{w} \leq v \\ \boldsymbol{w} \in \mathcal{C} \end{cases}$

$\boldsymbol{m} \equiv \mathrm{E}\left\{ \boldsymbol{R}_{T+\tau} \right\}$: expected returns

$\boldsymbol{S} \equiv \mathrm{Cov}\left\{ \boldsymbol{R}_{T+\tau} \right\}$: covariance

$$\boldsymbol{w}_v \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}' \widehat{\boldsymbol{m}} \right\}$$

subject to $\begin{cases} \boldsymbol{w}' \widehat{\boldsymbol{S}} \boldsymbol{w} \leq v \\ \boldsymbol{w} \in \mathcal{C} \end{cases}$

$\widehat{\boldsymbol{m}}$: estimate of $\boldsymbol{m}$

$\widehat{\boldsymbol{S}}$: estimate of $\boldsymbol{S}$

$\boldsymbol{w}$: relative portfolio weights

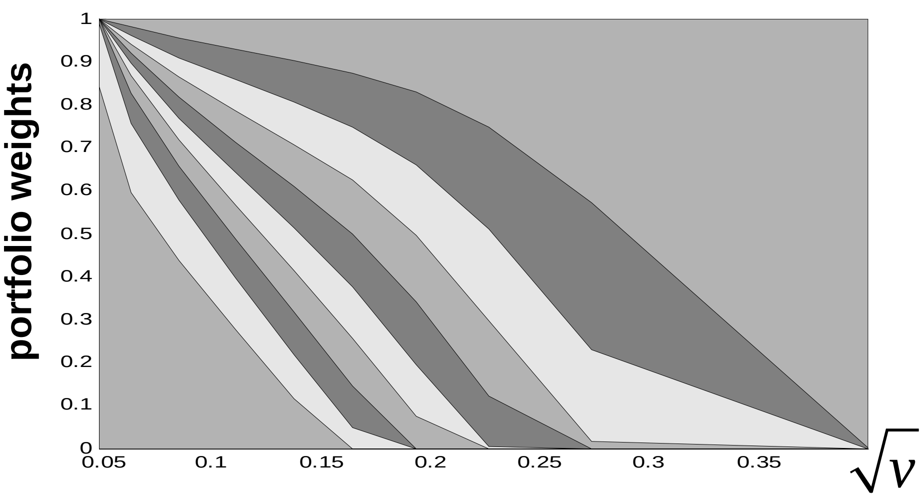
v : significant grid of target variances

$\mathcal{C}$: investment constraints, e.g. $\boldsymbol{w}' \boldsymbol{1} = 1, \ \boldsymbol{w} \geq \boldsymbol{0}$

ESTIMATION RISK – from true parameters…

$$\begin{cases} \boldsymbol{m} \equiv \delta \boldsymbol{S} \boldsymbol{w}_{eq} \\ \boldsymbol{S} \equiv \operatorname{diag}(\boldsymbol{s}) \begin{pmatrix} 1 & \rho & \rho & \cdots \\ \rho & 1 & \rho & \cdots \\ \rho & \rho & \ddots & \\ \vdots & \vdots & & 1 \end{pmatrix} \operatorname{diag}(\boldsymbol{s}) \end{cases}$$

$$\boldsymbol{w}_v \equiv \underset{\substack{\boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \leq v \\ \boldsymbol{w}'\boldsymbol{1}=1,\ \boldsymbol{w} \geq \boldsymbol{0}}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{m} \right\}$$



ESTIMATION RISK – … to estimates

$$r_t \sim \mathrm{N}(\boldsymbol{m}, \boldsymbol{S}), \quad t = 1, \ldots, T$$

1

$$\begin{cases} \boldsymbol{m} \equiv \delta \boldsymbol{S} \boldsymbol{w}_{eq} \\ \boldsymbol{S} \equiv \operatorname{diag}(\boldsymbol{s}) \begin{pmatrix} 1 & \rho & \rho & \cdots \\ \rho & 1 & \rho & \cdots \\ \rho & \rho & \ddots & \\ \vdots & \vdots & & 1 \end{pmatrix} \operatorname{diag}(\boldsymbol{s}) \end{cases}$$

$$\begin{cases} \widehat{\boldsymbol{m}} \equiv \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{r}_t \\ \widehat{\boldsymbol{S}} \equiv \frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{r}_t - \widehat{\boldsymbol{m}}\right)\left(\boldsymbol{r}_t - \widehat{\boldsymbol{m}}\right)' \end{cases}$$

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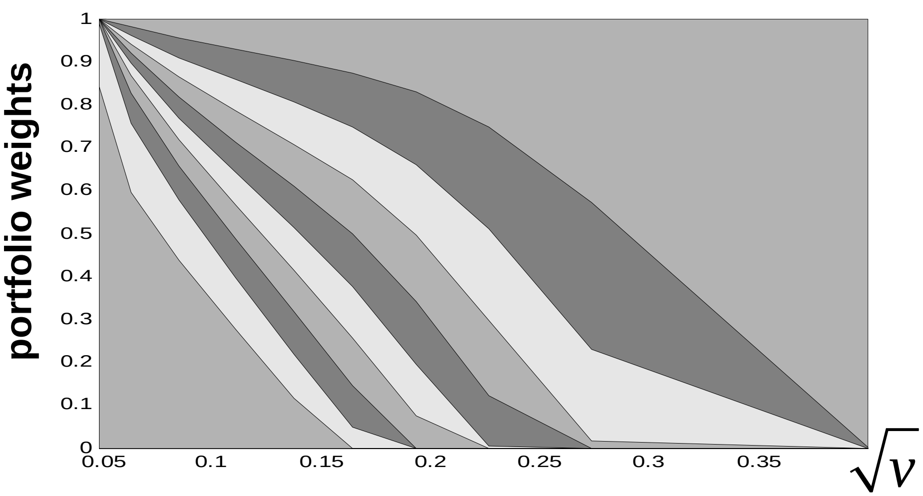

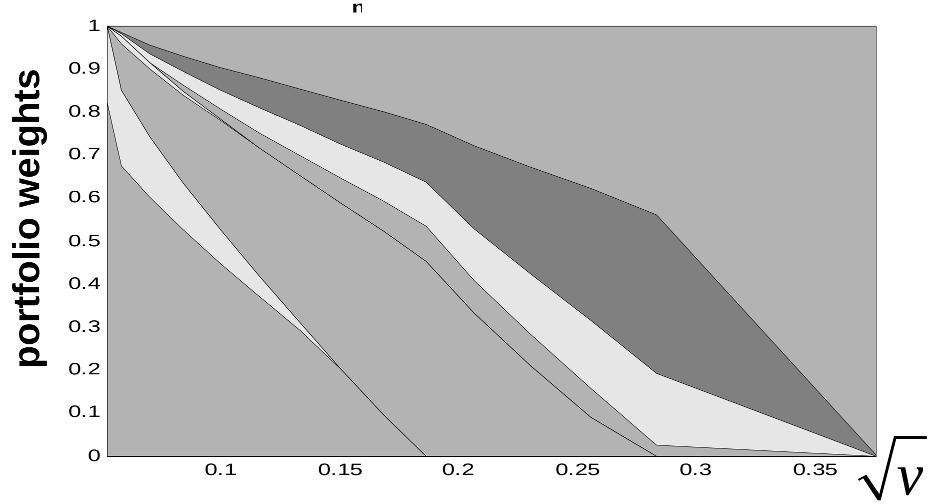

ESTIMATION RISK – … to estimates

$$r_t \sim \mathrm{N}(m, S), \quad t = 1, \ldots, T$$

2

$$\begin{cases} m \equiv \delta S w_{eq} \\ S \equiv \operatorname{diag}(s) \begin{pmatrix} 1 & \rho & \rho & \cdots \\ \rho & 1 & \rho & \cdots \\ \rho & \rho & \ddots & \\ \vdots & \vdots & & 1 \end{pmatrix} \operatorname{diag}(s) \end{cases}$$

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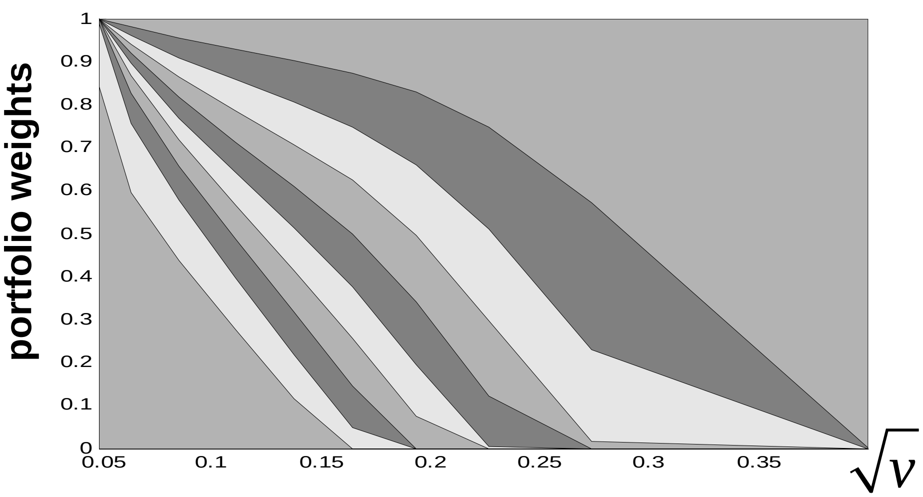

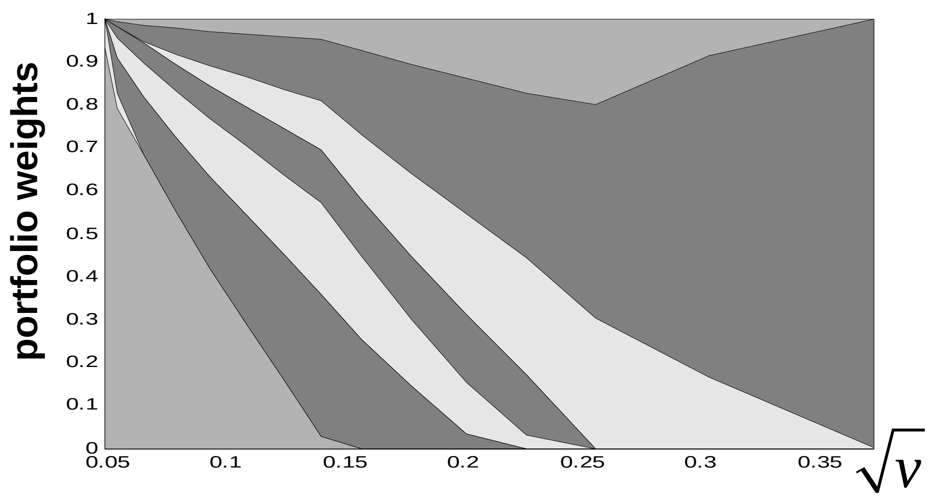

ESTIMATION RISK – … to estimates

3

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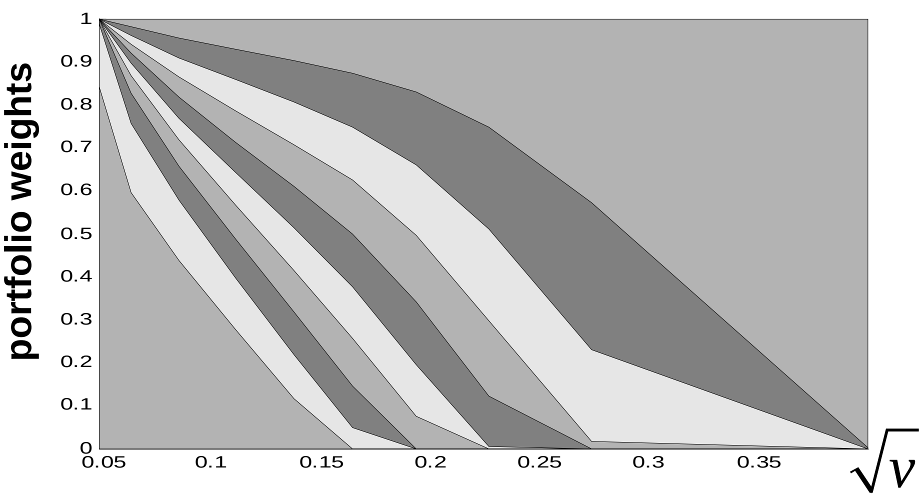

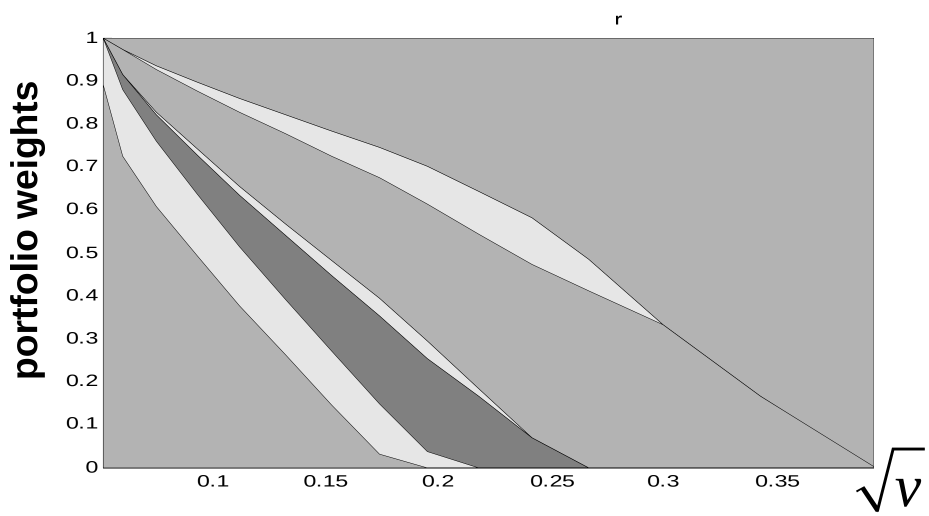

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$$w_v \equiv \underset{\substack{w'\widehat{S}w \leq v \\ w'1=1,\ w \geq 0}}{\operatorname{argmax}} \{ w'\widehat{m} \} \quad ? \; ? \; ?$$

Problem: uncertainty of input parameters

The exact value of the input parameters, i.e. expectations and covariances, is unknown. Different estimates give rise to different values for the input parameters that are close, but *not* equal to, the true, unknown parameters

ROBUST OPTIMIZATION

$$\boldsymbol{w}_v \equiv \underset{\substack{\boldsymbol{w}'\boldsymbol{S}\boldsymbol{w}\leq v \\ \boldsymbol{w}'\boldsymbol{1}=1,\ \boldsymbol{w}\geq\boldsymbol{0}}}{\operatorname{argmax}} \left\{\boldsymbol{w}'\boldsymbol{m}\right\}$$

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Problem: uncertainty of input parameters

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Solution: robust optimization

Optimize with respect to several estimates

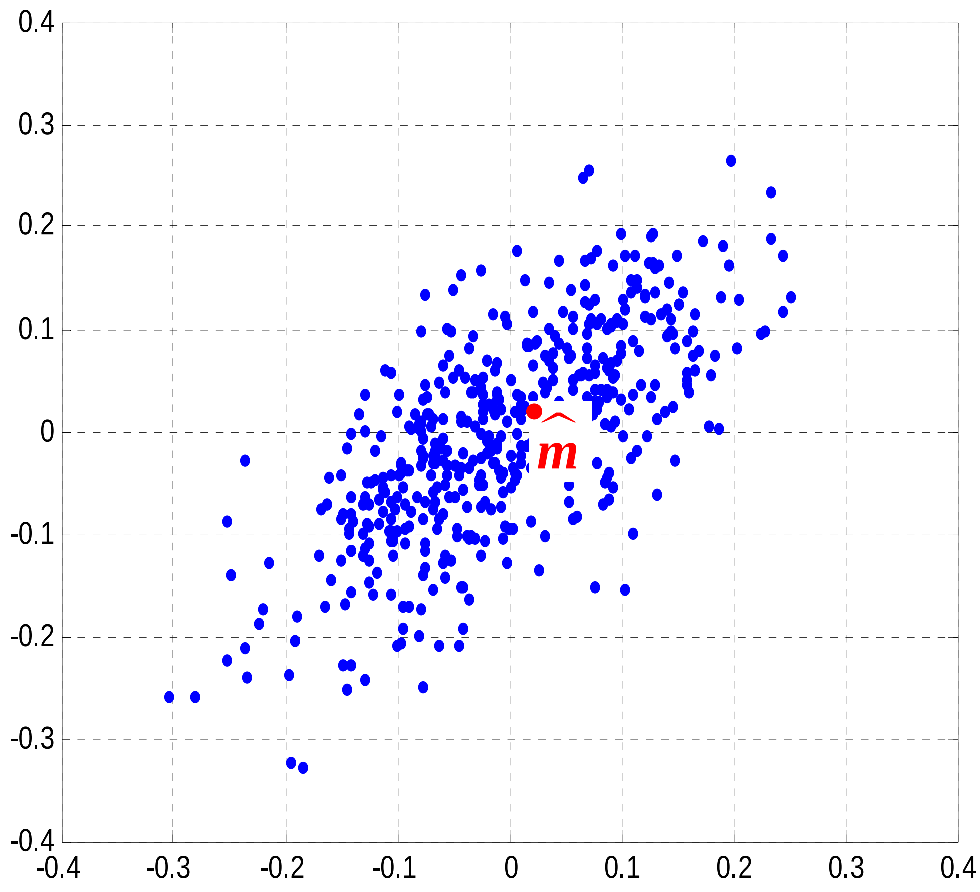
- conservative min-max
- Bayesian allocation

ROBUST OPTIMIZATION – uncertainty sets

$$\mathbf{r}_t \sim \mathrm{N}(\mathbf{m}, \mathbf{S}), \quad t = 1, \ldots, T$$

1

$$\widehat{\mathbf{m}} \equiv \frac{1}{T} \sum_{t=1}^{T} \mathbf{r}_t$$



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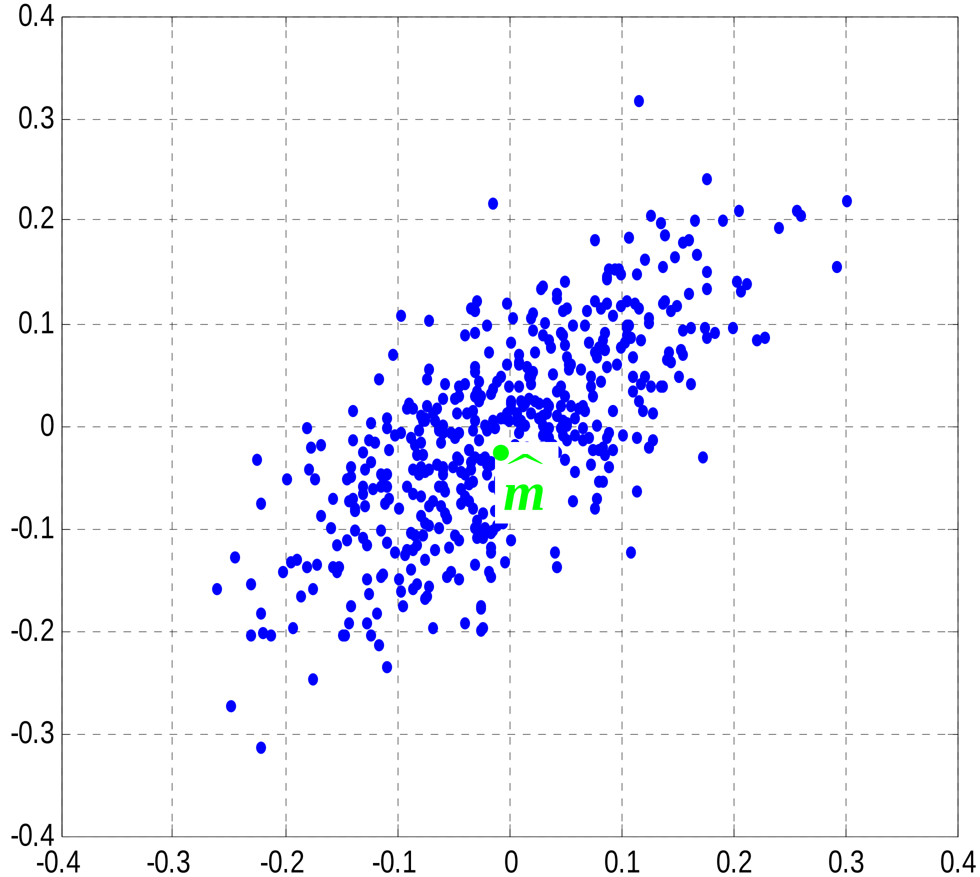

ROBUST OPTIMIZATION – uncertainty sets

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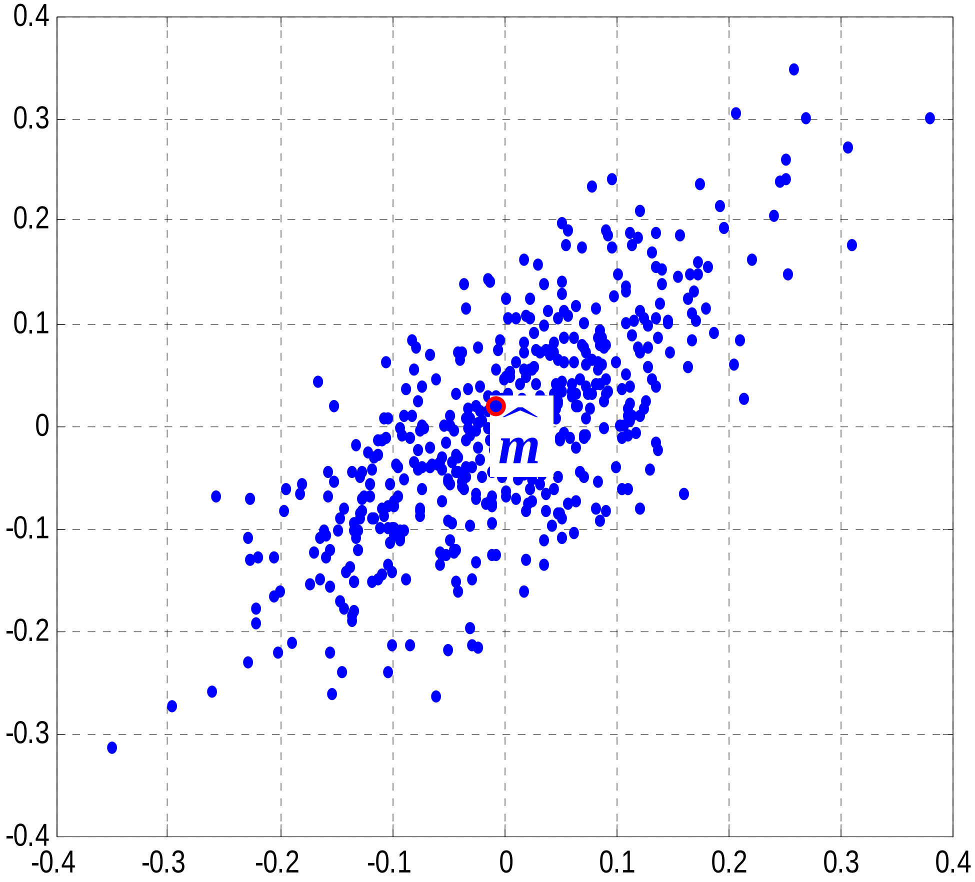
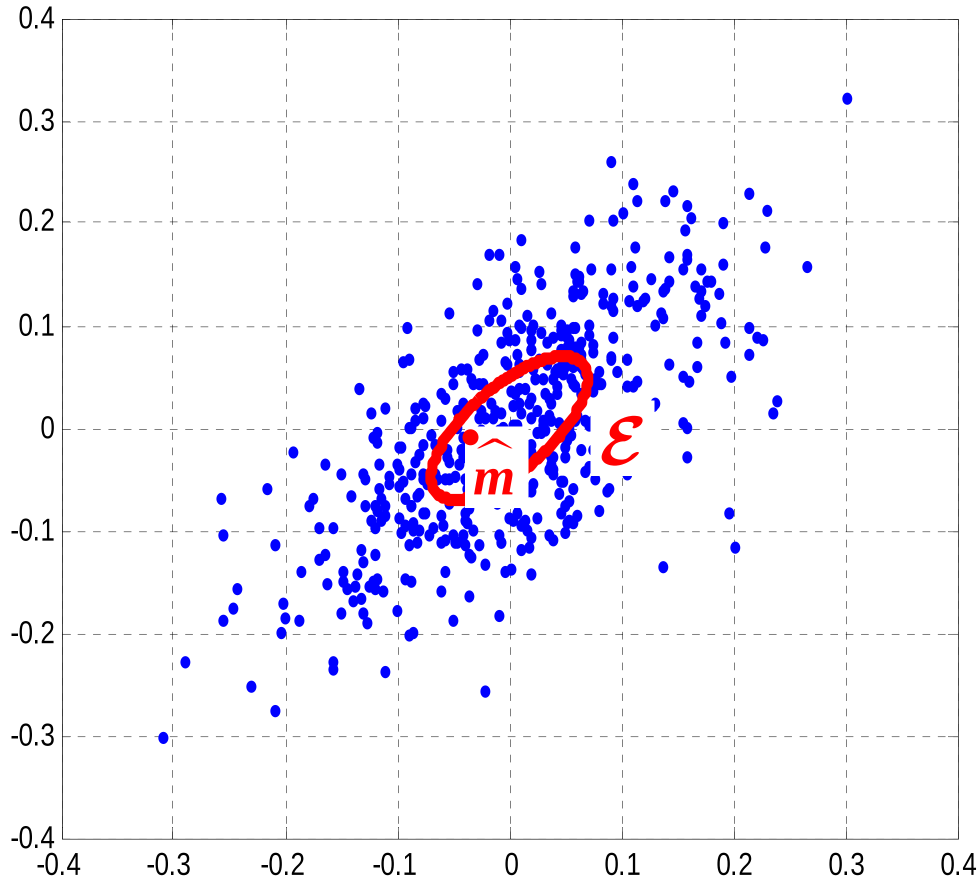


ROBUST OPTIMIZATION – uncertainty sets

$$\mathbf{r}_t \sim \mathrm{N}(\mathbf{m}, \mathbf{S}), \quad t = 1, \ldots, T$$

$$\widehat{\mathbf{m}} \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t \sim \mathrm{N}\left(\mathbf{m}, \frac{\mathbf{S}}{T}\right) \in \mathcal{E}$$



ROBUST OPTIMIZATION – conservative min-max approach

$$\boldsymbol{r}_t \sim \mathrm{N}(\boldsymbol{m}, \boldsymbol{S}), \quad t = 1, \ldots, T$$

$$\begin{cases} \widehat{\boldsymbol{m}} \equiv \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}_t \in \mathcal{E} \\ \widehat{\boldsymbol{S}} \equiv \frac{1}{T}\sum_{t=1}^{T} \left(\boldsymbol{r}_t - \widehat{\boldsymbol{m}}\right)\left(\boldsymbol{r}_t - \widehat{\boldsymbol{m}}\right)' \end{cases}$$

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$$\boldsymbol{w}_v \equiv \underset{\substack{\boldsymbol{w}'\widehat{\boldsymbol{S}}\boldsymbol{w} \leq v \\ \boldsymbol{w} \in \mathcal{C}}}{\operatorname{argmax}} \left\{ \min_{\widehat{\boldsymbol{m}} \in \mathcal{E}} \boldsymbol{w}'\widehat{\boldsymbol{m}} \right\}$$

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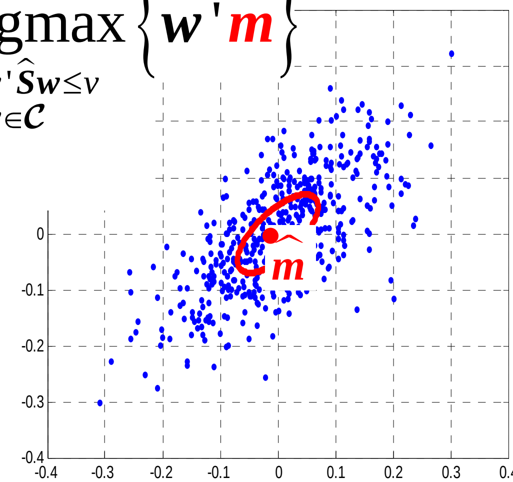

Problem: inefficient estimates

Estimates from one series and estimates from a different time series are completely different. The optimization process magnifies this discrepancy

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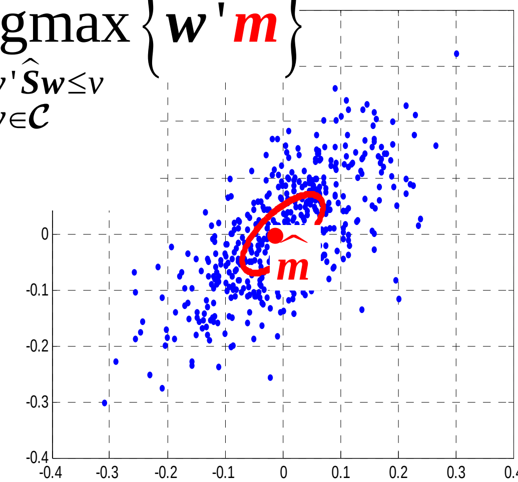

Problem: inefficient estimates

Estimates from one series and estimates from a different time series are completely different. The optimization process magnifies this discrepancy

Solution: shrinkage

Rely less on estimates, rely more on gut-feelings

$$\widehat{\boldsymbol{m}} \approx (1-c)\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{r}_t\right) + c\left(\boldsymbol{m}_0\right)$$

- Black-Litterman prior
- Rules of thumb
- Bayesian allocation

SHRINKAGE OPTIMIZATION – Black-Litterman

unconstrained mean-variance

$$\boldsymbol{w}_{\varsigma} \equiv \operatorname{argmax}\left\{\boldsymbol{w}'\boldsymbol{m} - \frac{1}{2\varsigma}\boldsymbol{w}'\boldsymbol{S}\boldsymbol{w}\right\}$$

$$\boldsymbol{w}_{\varsigma} \equiv \varsigma \boldsymbol{S}^{-1}\boldsymbol{m}$$

SHRINKAGE OPTIMIZATION – Black-Litterman

unconstrained mean-variance

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$$\mathbf{w}_{\varsigma} \equiv \varsigma \mathbf{S}^{-1}\mathbf{m}$$

$$\mathbf{m}_0 \equiv \frac{1}{\varsigma}\, \mathbf{S}\, \mathbf{w}_{\varsigma}$$

equilibrium portfolio

market-implied
equilibrium prior
expected returns

average risk
propensity ~ 0.4

exponentially
smoothed
estimate of
covariance

SHRINKAGE OPTIMIZATION – prior portfolio

constrained mean-variance

$$\boldsymbol{w}_{\varsigma} \equiv \underset{\boldsymbol{w} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}' \boldsymbol{m}_0 - \frac{1}{2\varsigma} \boldsymbol{w}' \boldsymbol{S} \boldsymbol{w} \right\}$$

preferred prior portfolio

average risk propensity ~ 0.4

exponentially smoothed estimate of covariance

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BAYESIAN OPTIMIZATION – Bayesian estimation theory

The Bayesian approach to estimation of the generic market parameters $\boldsymbol{m}$ differs from the classical approach in two respects:

- it blends historical information from time series analysis with experience
- the outcome of the estimation process is a (posterior) distribution, instead of a number

BAYESIAN OPTIMIZATION – Bayesian estimation theory

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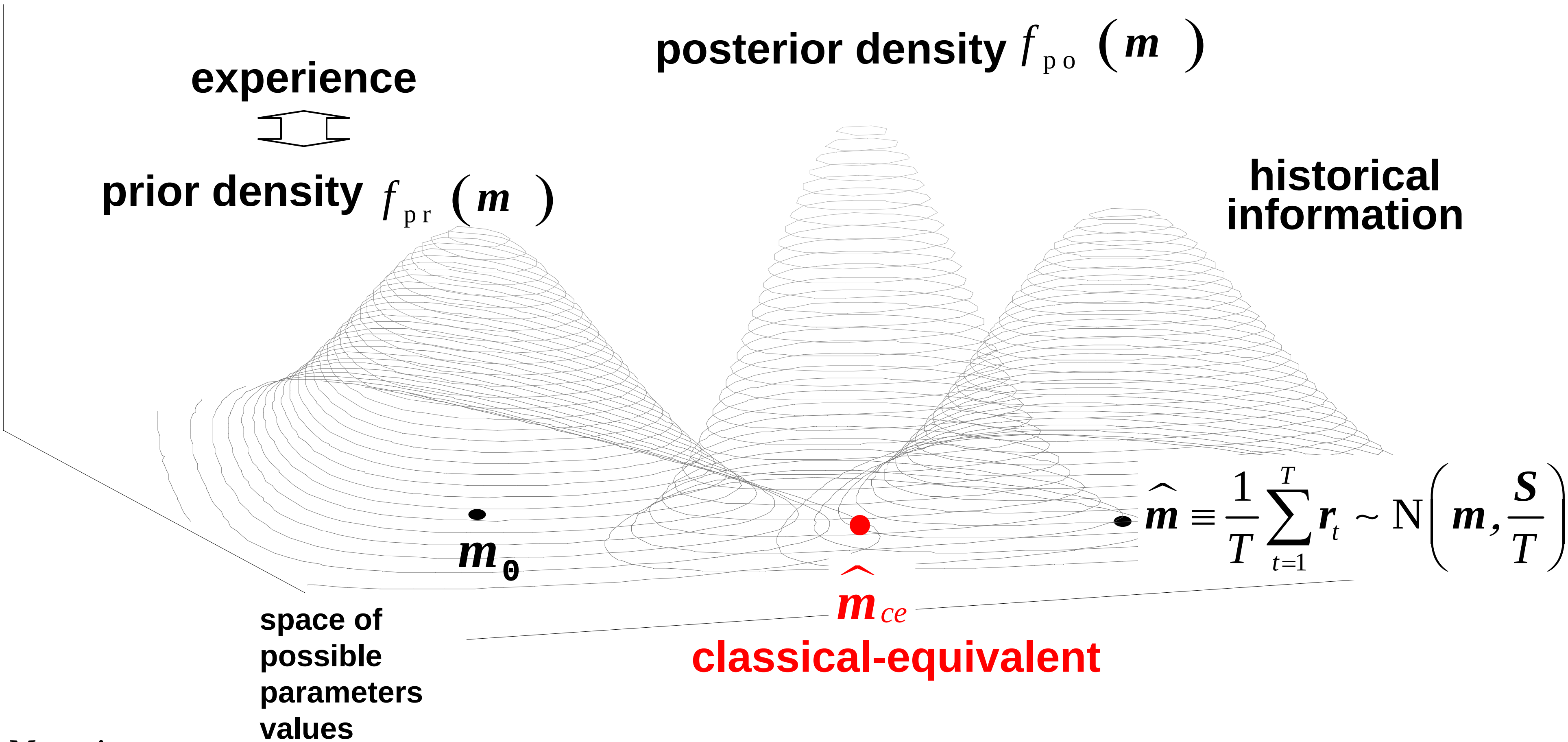
• it blends historical information from time series analysis with experience

• the outcome of the estimation process is a (posterior) distribution, instead of a number



BAYESIAN OPTIMIZATION – Bayesian estimation theory

theoretical optimization

$$\mathbf{w}_v \equiv \underset{\substack{\mathbf{w}'\mathbf{S}\mathbf{w} \leq v \\ \mathbf{w} \in \mathcal{C}}}{\operatorname{argmax}} \left\{ \mathbf{w}'\mathbf{m} \right\}$$

classical optimization

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BAYESIAN OPTIMIZATION – Bayesian estimation theory

theoretical optimization

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Bayesian optimization

$$\mathbf{w}_v \equiv \underset{\substack{\mathbf{w}'\widehat{\mathbf{S}}\mathbf{w} \leq v \\ \mathbf{w} \in \mathcal{C}}}{\operatorname{argmax}} \left\{ \mathbf{w}'\widehat{\mathbf{m}}_{ce} \right\} \qquad \widehat{\mathbf{m}}_{ce} \equiv (1-c)\left(\frac{1}{T}\sum_{t=1}^{T} \mathbf{r}_t\right) + c\left(\mathbf{m}_0\right)$$

Bayesian optimization is <u>shrinkage</u>

Bayesian optimization is (non-conservative) <u>robust</u>

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- This presentation:
  symmys.com > Teaching > Talks > *Handling Estimation Risk in Portfolio Optimization*

- implementation code (MATLAB):
  symmys.com > Book > Downloads > MATLAB

- Comprehensive discussion of

  - modeling
  - estimation
  - location-dispersion ellipsoid
  - satisfaction maximization
  - quantitative portfolio-management
  - risk-management
  - estimation risk
  - Black-Litterman allocation
  - Bayesian techniques
  - robust techniques
  - …

  symmys.com > Book > A. Meucci, *Risk and Asset Allocation* - Springer