# Valuation of Discrete Asian Options under Lévy processes

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# Abstract




- Aim: pricing of discrete monitored Asian options. This problem is usually reputed difficult for the path-dependent feature.

- We find an analytical solution for discrete monitored Geometric Asian options and propose a simple recursive procedure for Arithmetic Asian options.

- The dynamics of the underlying is described by a Lévy process, eventually allowing for stochastic volatility modelled by a stochastic time of change.

- The different Lévy models are calibrated to a set of plain vanilla options. Therefore, they share similar marginal distributions. What can be said about trajectory properties?

# Plan of the talk

- Lévy process
- Asian options: discrete and continuous monitoring.
- Analytical pricing of Geometric Asian options via Fourier Transform.
- Numerical pricing of Arithmetic Asian options via recursive integration.
- Computation of the Greeks.
- Numerical examples: monitoring frequency, model risk, effect of stochastic volatility.
- Conclusion.

# The Lévy process for the underlying



- We denote by $S_t$ the underlying asset price at time $t$, and let $X_t^\Delta$ to be the log-increment of size $\Delta$ :

$$X_t^\Delta \equiv \ln\left(S_{t+\Delta}\right) - \ln\left(S_t\right) \Longrightarrow S_{k\Delta} = s_0 e^{X_1^\Delta + \ldots + X_k^\Delta}$$

  and we assume that, under the risk neutral measure, $X_t^\Delta$ is a Levy process, i.e. a process which starts at zero and has independent and stationary increments.

- A generic Lévy process is fully determined by the characteristic function of its increments for a generic size $\Delta$,

$$\phi_{X^\Delta}(\omega) \equiv \mathbb{E}\left(e^{i\omega X_t^\Delta}\right).$$

In a equivalent way, a Lévy process is characterized by its characteristic exponent

$$\psi_\Delta(\omega) \equiv \ln\left(\phi_{X^\Delta}(\omega)\right).$$

# Lévy processes: examples



- In Table 1 we list a few parametric Levy processes

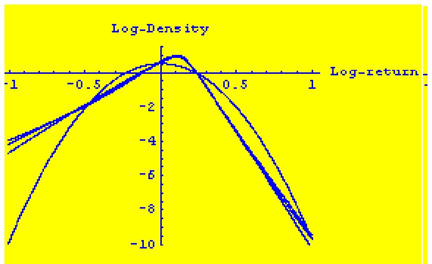
| Model | $\psi_\Delta(\omega)$ |
|---|---|
| Normal | $i\omega m\Delta - \frac{\sigma^2}{2}\omega^2\Delta$ |
| NIG | $i\omega m\Delta - \delta\Delta\left(\sqrt{\alpha^2-(\beta+i\omega)^2}-\sqrt{\alpha^2-\beta^2}\right)$ |
| VG | $i\omega m\Delta + C\Delta\ln\left(\frac{GM}{GM+(M-G)i\omega+\omega^2}\right)$ |
| Meixner | $i\omega m\Delta + 2\delta\Delta\ln\left(\frac{\cos(\beta/2)}{\cosh((\alpha\omega-i\beta)/2)}\right)$ |
| CGMY | $i\omega m\Delta + C\Delta\Gamma(-Y)\left((M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\right)$ |

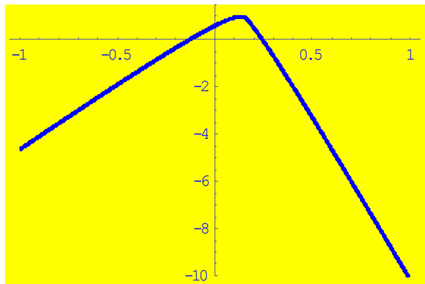
(1)

and $m$ is the value of the drift parameter such that the discounted price of the underlying is a martingale in the risk-neutral measure, see Schoutens (2003):

$$m = r - q - \ln\psi_\Delta(\omega)\big|_{\omega=-1,\Delta=1}.$$

- Why a Lévy process? It allows to generate skewed and fat distributions (smirk and smile implied volatility curves).



# Lévy process and Stochastic Volatility



- The main feature missing from the Lévy models described above is the fact that volatility (or more generally the environment) is changing stochastically over time.

- There are at least two ways of incorporating a volatility effect.

1. The first method makes the volatility parameter of the Black–Scholes model stochastic introducing a second stochastic differential equation, see Hull and White (1988) and Heston (1993).

2. The second way to build in stochastic volatility effects is to make time stochastic: in periods of high volatility, time will run faster than in periods of low volatility. This approach has been introduced by Clark (1973) and studied in detail by Geman and Ané (1996), Barndorff-Nielsen and Shephard (2001a,b, 2003b) and by Carr, Madan, Geman and Yor (2003).

# Characteristic function with Stochastic Volatility



- We follow the second approach, for wich a detailed account can be found in Carr, Geman, Madan and Yor (2003) and in Schoutens (2003). We write
$$S_{k\Delta} = s_0 e^{X^{\Delta}_{Y_1} + \ldots + X^{\Delta}_{Y_k}},$$
whith $Y_k$ representing the time change.

- If $Y_t$ is the stochastic time change and is characterized by the characteristic function $\varphi(\omega) = \mathbb{E}\exp(i\omega Y_t)$, then the log-return $X^{\Delta}_{Y_t}$ has characteristic function
$$\mathbb{E}e^{i\omega X^{\Delta}_{Y_t}} = e^{imt\omega}\varphi\left(-i\psi_{\Delta}(\omega)\right).$$

- A possible candidate for the rate of time change is given by the integrated CIR process $Y_t = \int_0^t y_s ds$, where:
$$\begin{aligned} dy_t &= \alpha(\eta - y_t)\,dt + \sigma\sqrt{Y_t}dW_t, y_0 = 1, \\ \varphi(\omega) &= \mathbb{E}\exp(i\omega Y_t) = \frac{e^{\alpha^2\eta t/\lambda^2} e^{2i\omega/(\alpha+\gamma\coth(\gamma t/2))}}{\left(\cosh(\gamma t/2) + \alpha\sinh(\alpha t/2)/\alpha\right)^{2\alpha\eta/\lambda^2}}, \\ \gamma &= \sqrt{\alpha^2 - 2\lambda^2 i\omega}. \end{aligned}$$

# Asian Options

- Asian options are written on an average: prices of an underlying security (or index) are recorded on a set of dates during the lifetime of the contract. At the option's maturity, a pay-off is computed as a deterministic function of an average of these figures.

- Asian options are quite popular among derivative traders and risk managers for several reasons. Primarily, Asian options smoothen possible market manipulations occurring near the expiry date. Moreover, these options provide a suitable hedge for firms facing a stream of positions. This is the case, for instance, of commodity end-users which are financially exposed to average prices.

- The standard Asian contract features an arithmetic average computed over weekly (or monthly) recorded observations. It is a common use to approximate this average with one on a continuum of values recorded over a time interval.

- In the standard Black-Scholes framework and continuous monitoring case there is the Geman-Yor (1993) analytical formula and several numerical approximations. For a comparison see Fusai and Roncoroni (2005).

- In the general Lévy setting, some new recent results by Albrecher et al. (2003, 2004) that propose approximations based on the moments or a semi-static hedging pricing procedure

# The option payoff



- At times $k\Delta$,$k = 1, ..., T$, we observe the stock price.
- Then we can define the geometric and arithmetic averages, $G_T^\Delta$ and $A_T^\Delta$

$$G_T^\Delta \equiv \left( \prod_{k=0}^{T} S_{k\Delta} \right)^{\frac{1}{T+1}} \text{ and } A_T^\Delta \equiv \frac{1}{T+1} \sum_{k=0}^{T} S_{k\Delta}. \tag{2}$$

- The payoff of an Asian call option with fixed strike $K$ reads:

$$\max \left\{ G_T^\Delta - K, 0 \right\} \text{ and } \max \left\{ A_T^\Delta - K, 0 \right\} \tag{3}$$

- Assuming continuous monitoring, i.e. $\Delta \to 0$ with $\Delta T = \tau$, and , we should define

$$G_\tau \equiv \exp \left( \frac{1}{\tau} \int_0^\tau \ln S_u du \right) \text{ and } A_\tau \equiv \frac{1}{\tau} \int_0^\tau S_u du. \tag{4}$$

# Pricing Geometric Asian options



- It is immediate to check the following equality for the log-geometric average:

$$\ln\left(G_T^\Delta\right) = \frac{1}{T+1}\sum_{t=0}^{T}\left(T-k+1\right)X_t^\Delta. \tag{5}$$

- Therefore, using the independence of each $X_t^\Delta$, we obtain the characteristic function of $\ln\left(G_T^\Delta\right)$:

$$\mathbb{E}\left(e^{i\omega\ln\left(G_T^\Delta\right)}\right) = \prod_{t=0}^{T}\mathbb{E}\left\{e^{i\omega\frac{T-t+1}{T+1}X_t^\Delta}\right\} = e^{\sum_{t=0}^{T} t\psi_\Delta\left(\omega\frac{T-k+1}{T+1}\right)}. \tag{6}$$

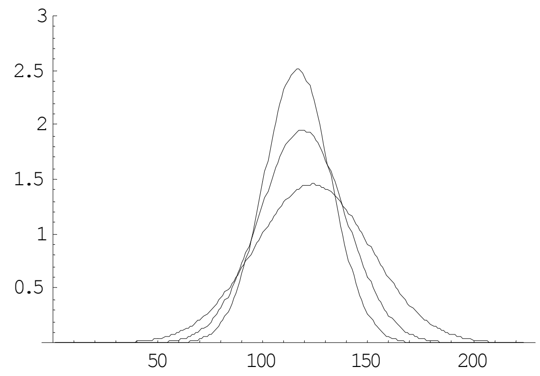
- We can easily compute numerically the inverse Fourier trannsform of this expression, obtaining the risk-neutral density of $\ln\left(G_T^\Delta\right)$ to be integrated over the option payoff, or, using the argument in Carr and Madan (1999), to give an expression for the Fourier transform of the call option price with respect to the logarithm of the strike

$$\int_{-\infty}^{+\infty} e^{i\omega k}e^{\alpha k}C\left(T,e^k\right)dk = \frac{e^{-rT}\mathbb{E}\left(e^{i(\omega-(\alpha+1)i)\ln\left(G_T^\Delta\right)}\right)}{\alpha^2+\alpha-\omega^2+i\left(2\alpha+1\right)\omega}.$$

- Notice that (6) is a new formula for pricing geometric-average Asian options under the very general assumption that the log-price of the underlying evolve according to a Lévy process. Only, for Levy stable processes $\ln\left(G_T^{\Delta}\right)$ has the same distribution as the individual returns $X_t^{\Delta}$.

# Pricing Geometric Asian options: the Algorithm

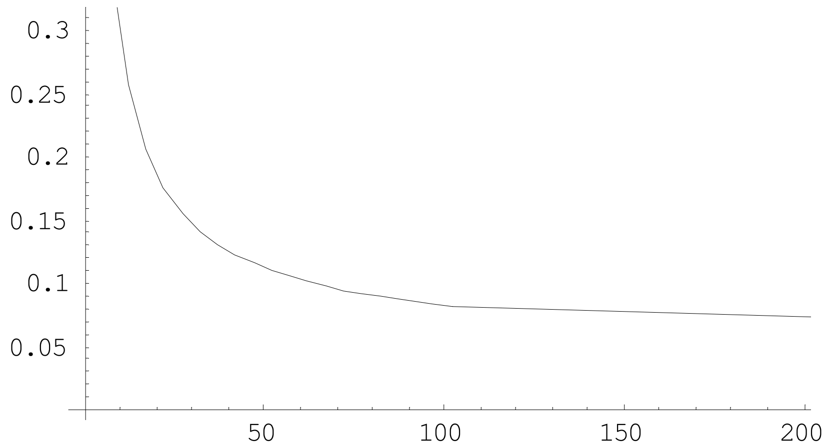
- Assign the characteristic function of the log-return, $\psi_\Delta(\omega)$.
- Compute the characteristic function of the geometric average.
- Using the FFT, find the density function or the option price.
- Eventually check the accuracy of your results comparing numerical and theoretical moments of $\ln\left(G_T^\Delta\right)$.



# Discrete and continuous Asian options

- The discrepancy between option prices under continuous and discrete monitoring can be huge.
- It does not exist an analytical formula for pricing discrete asian options, either in the Black-Scholes setting.

- The convergence of the discretely monitored option price to the continuous case is known to be extremely slow (order $1/\sqrt{n}$).




Price of the discrete barrier option versus number of monitoring dates.

# Example

Spot price=100, strike =100, barrier =98, $r$=0.10, $\sigma$=0.3, $T$=0.2.

| $n$ | Option Price | Mon. Frequency |
|---|---|---|
| 10 | 4.18224 | 5 gg |
| 50 | 3.12633 | 1 g |
| 100 | 2.86442 | 12h |
| 200 | 2.67640 | 6h |
| 500 | 2.50259 | 2h 24' |
| 1000 | 2.4125 | 1h 12' |
| 1500 | 2.37209 | 48' |
| 2000 | 2.35290 | 36' |
| 3000 | 2.18958 | 24' |
| $\infty$ | 2.18861 | cts |

# Pricing Arithmetic Asian options: a recursive formula

- As realized in Clelow and Carverhill (1990), a simple recursion can be found to obtain the distribution of the sum

$$\sum_{k=0}^{T} S_{k\Delta} = s_0 \left(1 + e^{X_1^\Delta} + e^{X_1^\Delta + X_2^\Delta} + + e^{X_1^\Delta + X_2^\Delta + \ldots + X_T^\Delta}\right)$$

- If we define recursively $B_0 = X_T^\Delta$ and

$$B_i = X_{T-i}^\Delta + \ln\left(1 + e^{B_{i-1}}\right), i = 1, ..., T$$

  then $\sum_{k=0}^{T} S_{k\Delta} = s_0 e^{B_T}$.

- By construction, $X_{T-i}^\Delta$ and $B_{i-1}$ are independent. Moreover, $X_i^\Delta$ are iid, Therefore the density of $B_i$ is the convolution of the density of $X_{T-i}^\Delta$, $\phi(x)$, and of $\ln\left(1 + e^{B_{i-1}}\right)$. With a change of variable, we can write a recursion on the density of $B_i$

$$\begin{aligned} f_{B_i}(x) &= \int_{-\infty}^{+\infty} \phi\left(x - \ln\left(e^y + 1\right)\right) f_{B_{i-1}}(y)\, dy, i = 1, ..., T, x \in R \\ f_{B_0}(x) &= \phi(x). \end{aligned}$$

# Pricing Arithmetic Asian options: quadrature formula



- We compute the integral using Gaussian quadrature

$$f_{B_i}(x) = \sum_{j=1}^{n} w_j \phi\left(x - \ln\left(e^{y_j} + 1\right)\right) f_{B_{i-1}}(y_j), j = 1, ...,$$

and considering the $n$ points $x_1, ..., x_n$

$$\mathbf{f}_i(\mathbf{x}) = \mathbf{A}\mathbf{f}_{i-1}(\mathbf{x}) \sum_{j=1}^{n} w_j \phi\left(x - \ln\left(e^{y_j} + 1\right)\right) f_{B_{i-1}}(y_j)$$

where the element $(k, j)$ of $A$ is $w_j \phi\left(x_j - \ln\left(e^{y_j} + 1\right)\right) f_{B_{i-1}}(y_j)$

- By free, we obtain also a recursion on the moments of the sum:

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=0}^{T} S_{k\Delta}\right)^n &= s_0^n \mathbb{E}\left(e^{nB_T}\right) \\
&= s_0^n \mathbb{E} e^{n\left(X_{T-i}^{\Delta} + \ln\left(1 + e^{B_{i-1}}\right)\right)} \\
&= s_0^n \mathbb{E} e^{nX^{\Delta}} \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}\left(e^{jB_{T-1}}\right),
\end{aligned}$$

and

$$\mathbb{E}\left(e^{nB_T}\right) = s_0^n \mathbb{E} e^{nX^{\Delta}} \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}\left(e^{jB_{T-1}}\right),$$

recursion that requires the knowledge of the characteristic function of $X^{\Delta}$, always available in our case.

This recursion has been exploited in Albrecher to get approximate formulae for Arithmetic options. The above moment recursion will provide to be useful in order to verify the accuracy of the method.

# Conclusions and extensions

- In this paper we have proposed an analytical and numerical methods to price discrete asian options.
- The procedure requires that the kernel if of convolution type and its Fourier transform is known (e.g. Lévy processes).
- The idea can be applied to barrier options and to the continuos monitoring case as well.

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