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# Toda Equations, bi-Hamiltonian Systems, and Compatible Lie Algebroids

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**Abstract.** We present the bi-Hamiltonian structure of $\text{Toda}_3$, a dynamical system studied by Kupershmidt as a restriction of the discrete KP hierarchy. We derive this structure by a suitable reduction of the set of maps from $\mathbb{Z}_d$ to $\text{GL}(3, \mathbb{R})$, in the framework of Lie algebroids.

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**Key words:** Toda lattice, Lie algebroids, bi-Hamiltonian manifolds, Marsden–Ratiu reduction.

## 1. Introduction

It is well known (see [12] and references therein) that the periodic Toda lattice is a bi-Hamiltonian system and that its integrability properties can be easily derived by its bi-Hamiltonian structure. In [11], this structure is investigated by means of a new approach: it stems from a reduction process of a special kind of Lie algebroids [6]. This approach parallels the work of [2], where the 'continuous counterpart' of the Toda lattice is studied, namely, the KdV equation.

Indeed, the KdV equation is a bi-Hamiltonian system obtained by reducing the space $\text{Map}(S^1, \mathfrak{gl}(2, \mathbb{R}))$ of $\mathcal{C}^\infty$ maps from $S^1$ to $\mathfrak{gl}(2, \mathbb{R})$. If instead of the space $\text{Map}(S^1, \mathfrak{gl}(2, \mathbb{R}))$ one considers the space $\text{Map}(S^1, \mathfrak{gl}(3, \mathbb{R}))$, one obtains the Boussineq hierarchy, which also displays a bi-Hamiltonian structure. The discrete version of the KdV equation, the Toda lattice, is obtained in [11] by replacing the circle $S^1$ with the cyclic group $\mathbb{Z}_d$ and the algebra $\mathfrak{gl}(2, \mathbb{R})$ with the group $\text{GL}(2, \mathbb{R})$: therefore, the space to reduce is $\text{Map}(\mathbb{Z}_d, \text{GL}(2, \mathbb{R}))$.

In this paper we analyze the discrete version of the Boussineq equation. We consider therefore the reduction of the space $\text{Map}(\mathbb{Z}_d, \text{GL}(3, \mathbb{R}))$. The equations we obtain also display a bi-Hamiltonian structure, which, to the best of our knowledge, is not known in the literature. We call $\text{Toda}_3$ the integrable dynamical system that arises naturally from this structure. This dynamical system is studied under a different perspective by Kupershmidt in [5].

The plan of the paper is the following. In Section 2 we introduce the geometrical structure of the phase space to reduce: the set of maps $\text{Map}(\mathbb{Z}_d, \text{GL}(3, \mathbb{R}))$. Following the analysis in [11], this space can be endowed with the structure of a *Poisson bi-anchored manifold*. In Section 3 we present the reduction of the Poisson bi-anchored manifold $\mathrm{Map}(\mathbb{Z}_d, \mathrm{GL}(3, \mathbb{R}))$. This reduction is an adaptation of the Marsden–Ratiu reduction scheme for Poisson manifolds [9] and gives rise to a bi-Hamiltonian manifold. In Section 4 we apply the theory of Gelfand and Zakharevich [4] to study the integrability properties of the bi-Hamiltonian flows obtained previously. The last section contains an example.

## 2. Poisson Bi-anchored Manifolds

In this section we introduce the geometric objects we need for our approach to the Toda lattice. The discussion closely follows [11]. We will endow the manifold $\mathcal{M}$ of the maps from the cyclic group $\mathbb{Z}_d$ to $\mathrm{GL}(3, \mathbb{R})$ with several structures, namely a Poisson tensor and two compatible Lie algebroids (see [6]) suitably soldered together: this will make $\mathcal{M}$ into a Poisson bi-anchored manifold.

First we introduce the manifold $\mathcal{M}$. A point $q$ of $\mathcal{M}$ is simply a $d$-tuple of invertible $3 \times 3$ matrices

$$q = \left(q^1, \ldots, q^d\right), \tag{1}$$

where

$$q^k = \begin{pmatrix} q_1^k & q_2^k & q_3^k \\ q_4^k & q_5^k & q_6^k \\ q_7^k & q_8^k & q_9^k \end{pmatrix}. \tag{2}$$

We will always be dealing with $d$-tuples of matrices and the following condition is supposed to hold throughout the discussion:

$$(\cdot)^{k+d} = (\cdot)^k. \tag{3}$$

For convenience, we will say that a matrix as in (2) represents a $d$-tuple as in (1). Vector fields on $\mathcal{M}$ will be represented by $d$-tuples of $3 \times 3$ matrices $\dot{q}^k$ whose entries are functions of the point $q \in \mathcal{M}$. The same way, one-forms on $\mathcal{M}$ are represented as $d$-tuples of $3 \times 3$ matrices $\alpha^k$ where each entry is a function of the point. The value of the one-form $\alpha$ on the vector field $\dot{q}$ is given by the scalar function:

$$\langle \alpha, \dot{q} \rangle = \sum_{k=1}^{d} \mathrm{Tr}\left(\alpha^k \dot{q}^k\right). \tag{4}$$

Next, we endow $\mathcal{M}$ with a Poisson manifold structure. A quick computation shows that the map $P' \colon T^*\mathcal{M} \to T\mathcal{M}$ defined as

$$\dot{q}^k = P'(\alpha)^k = q^k \alpha^k b - b \alpha^k q^k, \tag{5}$$

where $b$ is any fixed matrix, is indeed a Poisson tensor, i.e., it defines a Poisson bracket $\{f, g\} = \langle \mathrm{d}f, P'\mathrm{d}g \rangle$. In order to recover the Toda lattice we choose

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

To obtain the Hamiltonian vector field $X_H$ associated with a (Hamiltonian) function $H\colon \mathcal{M} \to \mathbb{R}$, we simply have to plug its differential $\alpha = \mathrm{d}H$ into Equation (5).

At this point we endow $\mathcal{M}$ with an additional structure: a pencil of Lie algebroids. We recall (see [6]) that $(\mathcal{M}, \mathcal{E}, A, \{\cdot, \cdot\})$ is a *Lie algebroid* if

(i) $\mathcal{E}$ is a vector bundle on $\mathcal{M}$
(ii) $\{\cdot, \cdot\}$ is a bilinear composition law on the sections of $\mathcal{E}$ that makes them into a Lie algebra
(iii) the map $A\colon \mathcal{E} \to T\mathcal{M}$, called an anchor, is a Lie algebra morphism:

$$A(\{s, t\}) = [A(s), A(t)], \tag{7}$$

where $[\cdot, \cdot]$ is the usual commutator of vector fields.

If two different Lie algebroid structures $(\mathcal{M}, \mathcal{E}, A, \{\cdot, \cdot\})$ and $(\mathcal{M}, \mathcal{E}, A^{'}, \{\cdot, \cdot\}^{'})$ coexist on the same manifold $\mathcal{M}$ and vector bundle $\mathcal{E}$ we can consider the pencil of brackets

$$\{s, t\}_\lambda = \{s, t\} + \lambda \{s, t\}^{'} \tag{8}$$

and the pencil of maps

$$A_\lambda(s) = A(s) + \lambda A^{'}(s), \tag{9}$$

where $\lambda$ is a complex parameter. The two Lie algebroid structures are said to be compatible if $(\mathcal{M}, \mathcal{E}, A_\lambda, \{\cdot, \cdot\}_\lambda)$ is a Lie algebroid for every value of $\lambda$.

In our case, we consider the trivial vector bundle $\mathcal{E} = \mathcal{M} \times [\mathrm{Mat}(3, \mathbb{R})]^d$, where $(\cdot)^d$ denotes the $d$-times Cartesian product $(\cdot) \times \cdots \times (\cdot)$. The sections of this bundle are represented by $d$-tuples of $3 \times 3$ matrices $s^k$ whose entries are functions of the point $q \in \mathcal{M}$. Then we define the pencil of anchors $A_\lambda\colon \mathcal{E} \to T\mathcal{M}$ as

$$\dot{q}^k = A_\lambda(s)^k = s^{k+1}\big(q^k + \lambda b\big) - \big(q^k + \lambda b\big)s^k, \tag{10}$$

and a composition law $\{\cdot, \cdot\}_\lambda$ as

$$\{s, t\}_\lambda^k = \partial_{A(s)} t^k - \partial_{A(t)} s^k + \big[t^k, s^k\big] + \lambda\big(\partial_{A'(s)} t^k - \partial_{A'(t)} s^k\big), \tag{11}$$

where by $\partial_{\dot{q}} t$ we mean the derivative of the section $t$ along the vector field $\dot{q}$. It is easy to prove that the manifold $(\mathcal{M}, \mathcal{E}, A_\lambda, \{\cdot, \cdot\}_\lambda)$ is a pencil of Lie algebroids.

The pencil of Lie algebroids structure defines a useful relation among one-forms. We define a one-form $\alpha$ to be related with a one-form $\beta$ (and we denote this by $\alpha \sim \beta$) if $A^*\alpha = A'^*\beta$. In order to explicitly calculate this relation we need to derive the expression of the dual pencil of anchors $A^*_\lambda = A^* + \lambda A'^*$. An element $\xi$ of the dual vector bundle $\mathcal{E}^*$ can be naturally identified with a point of $\mathcal{E}$ by means of the pairing

$$\langle \xi, s\rangle = \sum_{k=1}^{d} \mathrm{Tr}\big(\xi^k s^k\big).$$

Therefore the dual pencil $A^*_\lambda\colon T^*\mathcal{M} \to \mathcal{E}^*$ can be viewed as a map that with a one-form $\alpha$ associates a $d$-tuple $\xi^k$ of matrices whose entries are functions of the point $q \in \mathcal{M}$. A quick calculation shows that $A^*_\lambda$ reads:

$$\xi^k = A^*_\lambda(\alpha)^k = \big(q^{k-1} + \lambda b\big)\alpha^{k-1} - \alpha^k\big(q^k + \lambda b\big). \tag{12}$$

Thus far we have defined a Poisson structure and a pencil of Lie algebroids on the manifold $\mathcal{M}$. We arrived at the last step: soldering these structures by means of two maps $J, J'\colon T^*\mathcal{M} \to \mathcal{E}$ that verify the following conditions: if $\alpha \sim \beta$, then

$$P'(\alpha) = A'(J\alpha + J'\beta), \tag{13}$$

$$P'(\beta) = A(J\alpha + J'). \tag{14}$$

It is easy to see that the intertwining maps defined as

$$J\colon s^k = \alpha^k q^k, \qquad J'\colon s^k = -\alpha^k b,$$

satisfy the above relation. This ends the definition of the geometrical structures of the manifold $\mathcal{M} = \mathrm{Map}(\mathbb{Z}_d, \mathrm{GL}(3,\mathbb{R}))$: it is a Poisson manifold endowed with two compatible Lie algebroid structures that define a relation on one-forms and two intertwining maps that solder everything. We call such a structure a Poisson bi-anchored manifold.

## 3. The Reduction

In this section we perform the reduction of the manifold $\mathcal{M} = \mathrm{Map}(\mathbb{Z}_d, \mathrm{GL}(3,\mathbb{R}))$ introduced in the previous section. Combining a restriction and a projection we obtain a new manifold $\mathcal{N}$ of lower dimension endowed with the same geometrical structure as the original manifold $\mathcal{M}$, but with the important additional property of being bi-Hamiltonian.

As a first step, we consider the distribution $\mathrm{Im}\, P' + \mathrm{Im}\, A'$. It is easy to verify by formulas ( 5) and (10) that this distribution is integrable and therefore it foliates $\mathrm{Map}(\mathbb{Z}_d, \mathrm{GL}(3,\mathbb{R}))$ in maximal integral leaves, which are the $5d$-dimensional hyperplanes of the form

$$q^k = \begin{pmatrix} q_1^k & q_2^k & q_3^k \\ q_4^k & v_5^k & v_6^k \\ q_7^k & v_8^k & v_9^k \end{pmatrix}, \tag{15}$$

where the $\nu_i^k$ are constants. The restriction process we mentioned above consists in selecting one of these leaves. To obtain the Toda$_3$ system, we pick the leaf $\mathcal{L}$ defined by points of the form

$$q^k = \begin{pmatrix} q_1^k & q_2^k & q_3^k \\ q_4^k & 0 & 0 \\ q_7^k & 1 & 0 \end{pmatrix}. \tag{16}$$

The pencil of anchors allows us to define another distribution, namely $D = A(\ker A')$, which is integrable.⋆ As opposed to the $\mathrm{GL}(2, \mathbb{R})$ case, this distribution is not tangent to $\mathcal{L}$. We are interested only in the restriction of this distribution to the leaf $\mathcal{L}$, which we denote by $D|_{\mathcal{L}}$. The distribution $E = D|_{\mathcal{L}} \cap T\mathcal{L}$ of $\mathcal{L}$ is also integrable and an explicit computation shows that $E$ is spanned by the vector fields of the form

$$\begin{array}{lllll} \dot{q}_1^k = 0, & \dot{q}_2^k = \mu^k q_2^k - q_3^k s_8^k, & \dot{q}_3^k = \mu^{k-1} q_3^k, & \dot{q}_4^k = -\mu^{k+1} q_4^k, & \\ \dot{q}_5^k = 0, & \dot{q}_6^k = 0, & \dot{q}_7^k = \tau^k q_4^k - \mu^k q_7^k, & \dot{q}_8^k = 0, & \dot{q}_9^k = 0 \end{array} \tag{17}$$

for arbitrary $\mu^k, \tau^k$. From this expression we see that along the vector fields in $E$ the following equations are satisfied:

$$\dot{q}_1^k = 0, \qquad \left(q_2^{k+1} q_4^k + q_3^{k+1} q_7^k\right)^\bullet = 0, \qquad \left(q_4^k q_3^{k+2}\right)^\bullet = 0.$$

This means that the distribution $E$ admits the three invariants

$$a_1^k = q_1^k, \tag{18}$$
$$a_2^k = q_2^{k+1} q_4^k + q_3^{k+1} q_7^k, \tag{19}$$
$$a_3^k = q_4^k q_3^{k+2}. \tag{20}$$

At this point we can operate the projection we mentioned above: we define the reduced manifold $\mathcal{N}$ to be the quotient of the leaf $\mathcal{L}$ with respect to the foliation induced by the distribution $E$. By (18), (19) and (20) we argue that $\mathcal{N}$ is a $3d$-dimensional manifold that can be regarded as $\mathbb{R}^{3d}$, endowed with the set of coordinates $(a_1^k, a_2^k, a_3^k)_{k=1,\dots,d}$. The above formulas also yield the expression of the canonical projection $\pi\colon \mathcal{L} \to \mathcal{N}$.

After obtaining the reduced manifold $\mathcal{N}$ we endow it with a Poisson structure. This can be done observing that $(\mathcal{M}, P', D, \mathcal{L})$ is Poisson reducible, in the terminology of [9]. In this context, to find the expression of the reduction of $P'$ we have to extend a generic one-form $\varphi$ on $\mathcal{N}$ to a one-form $\alpha$ on $\mathcal{M}$ (possibly defined only at the points of the leaf $\mathcal{L}$) which annihilates the distribution $D$. This means:

$$\langle \alpha, D \rangle = 0, \qquad \langle \alpha, \dot{q} \rangle = \langle \varphi, \pi_* \dot{q} \rangle.$$

Let us denote by $\alpha = \mathrm{ext}(\varphi)$ any such extension. Then the expression

$$\mathfrak{p}'(\varphi) = \pi_* \circ P'(\mathrm{ext}(\varphi)) \tag{21}$$

⋆ This is true in general, provided that for any two sections $s, t$ such that $A'(s) = 0$ and $A'(t) = 0$ we have $\{s, t\}' = 0$. It is evident from (11) that this condition holds in our case.

does not depend on the choice of $\mathrm{ext}(\varphi)$ and determines a Poisson structure on $\mathcal{N}$. If we denote by $\varphi = \sum_{k=1}^{d} \varphi_1^k \mathrm{d}a_1^k + \varphi_2^k \mathrm{d}a_2^k + \varphi_3^k \mathrm{d}a_3^k$ the generic one-form on $\mathcal{N}$, an easy calculation shows that an extension $\alpha = \mathrm{ext}(\varphi)$ has the form

$$\alpha^k = \begin{pmatrix} \varphi_1^k & \varphi_3^k q_3^{k+2} + q_2^{k+1}\varphi_2^k & \varphi_2^k q_3^{k+1} \\ \varphi_2^{k-1} q_4^{k-1} & \alpha_5^k & q_3^{k+1}\varphi_3^{k-1} q_4^{k-1} \\ \varphi_3^{k-2} q_4^{k-2} + q_7^{k-1}\varphi_2^{k-1} & \alpha_8^k & \alpha_9^k \end{pmatrix}, \tag{22}$$

where

$$\alpha_9^k = q_7^{k-1}\varphi_3^{k-1} q_3^{k+1} - q_2^k \varphi_3^{k-2} q_4^{k-2} + \alpha_5^{k-1}.$$

As we said, this matrix is not completely determined: the components $\alpha_5^k, \alpha_8^k$ are free, since the extension gives an equivalence class of one-forms. Now we can apply formula (21) to obtain the expression of the reduced Poisson tensor $\mathfrak{p}'$:

$$\begin{aligned} \dot{a}_1^k &= \varphi_2^{k-1} a_2^{k-1} - \varphi_2^k a_2^k + \varphi_3^{k-2} a_3^{k-2} - \varphi_3^k a_3^k, \\ \dot{a}_2^k &= \left(\varphi_1^k - \varphi_1^{k+1}\right) a_2^k - \varphi_2^{k+1} a_3^k + \varphi_2^{k-1} a_3^{k-1}, \\ \dot{a}_3^k &= \left(\varphi_1^k - \varphi_1^{k+2}\right) a_3^k. \end{aligned} \tag{23}$$

Recalling the periodic Toda case [11], to find another Poisson structure we have to determine the reduced relation on the one-forms of $\mathcal{N}$. Therefore we need the expression of the reduced dual pencil of anchors $\mathfrak{a}_\lambda^*$. To have this, in turn, we have to define a proper reduced vector bundle $\mathcal{U}$ based on $\mathcal{N}$ on which the reduced anchors act: $\mathfrak{a}_\lambda \colon \mathcal{U} \to T\mathcal{N}$. It is convenient to define first the dual vector bundle $\mathcal{U}^*$ and the dual pencil $\mathfrak{a}_\lambda^*$. The definition of the vector bundle $\mathcal{U}$ and the pencil $\mathfrak{a}_\lambda$ will then follow by duality. A lenghty computation [10] allows us to find all these characters. We are only interested in the reduced relation on one-forms, which turns out to be the following: $\varphi \sim \psi$ (i.e., $\mathfrak{a}^*(\varphi) = \mathfrak{a}'^*(\psi)$) if and only if they satisfy

$$\begin{aligned} \psi_1^k - \psi_1^{k+1} \; = \; & \varphi_1^k a_1^k - \varphi_1^{k+1} a_1^{k+1} + \varphi_2^{k-1} a_2^{k-1} - \varphi_2^{k+1} a_2^{k+1} + \\ & + \varphi_3^{k-2} a_3^{k-2} - \varphi_3^{k+1} a_3^{k+1}, \end{aligned} \tag{24}$$

$$\begin{aligned} & -\psi_2^{k-2} a_3^{k-2} + \psi_2^k a_3^{k-1} \\ & \quad = \varphi_1^{k-2} a_3^{k-2} - \varphi_1^{k+1} a_3^{k-1} - \varphi_2^{k-2} a_3^{k-2} a_1^{k-1} + \\ & \qquad + \varphi_2^k a_3^{k-1} a_1^k - \varphi_3^{k-2} a_3^{k-2} a_2^{k-1} + \varphi_3^{k-1} a_3^{k-1} a_2^{k-1}, \end{aligned} \tag{25}$$

$$\begin{aligned} & \psi_2^k a_2^k - \psi_2^{k-1} a_2^{k-1} + \psi_3^k a_3^k - \psi_3^{k-2} a_3^{k-2} \\ & \quad = \varphi_1^{k-1} a_2^{k-1} - \varphi_1^{k+1} a_2^k + \\ & \qquad + \varphi_2^{k-2} a_3^{k-2} - \varphi_2^{k+1} a_3^k + a_1^k\left(\varphi_2^k a_2^k - \varphi_2^{k-1} a_2^{k-1}\right) + a_1^k\left(\varphi_3^k a_3^k - \varphi_3^{k-2} a_3^{k-2}\right). \end{aligned} \tag{26}$$

Now we focus on the special feature of the *reduced* Poisson bi-anchored manifold $\mathcal{N}$. By Equations (24), (25) and (26) for a fixed one-form $\varphi$ there is a whole class of one-forms $[\psi] = \psi + \ker \mathfrak{a}'^*$ that is related with it. In the reduced structure, though, $\ker \mathfrak{a}'^* \subset \ker \mathfrak{p}'$. Therefore the following tensor,

$$\mathfrak{p}(\varphi) := \mathfrak{p}'(\psi), \tag{27}$$

is well defined. A lengthy calculation shows that the bracket induced by this tensor verifies the Jacobi identity. Furthermore, $\mathfrak{p}$ is compatible with $\mathfrak{p}'$, i.e., the pencil $\mathfrak{p}_\lambda = \mathfrak{p} + \lambda\mathfrak{p}'$ is a Poisson tensor* for all values of the complex parameter $\lambda$. Explicitly, $\dot{a} = \mathfrak{p}(\varphi)$ reads

$$\begin{aligned}
\dot{a}_1^k &= a_1^k\big(\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k + \varphi_3^{k-2}a_3^{k-2} - \varphi_3^k a_3^k\big) + \\
&\quad + a_2^k\varphi_1^{k+1} - a_2^{k-1}\varphi_1^{k-1} + a_3^k\varphi_2^{k+1} - a_3^{k-2}\varphi_2^{k-2}, \\
\dot{a}_2^k &= a_1^k\big(\varphi_1^k a_2^k + \varphi_2^{k-1}a_3^{k-1}\big) - \\
&\quad - a_1^{k+1}\big(\varphi_1^{k+1}a_2^k + \varphi_2^{k+1}a_3^k\big) + \varphi_1^{k+2}a_3^k - \varphi_1^{k-1}a_3^{k-1} + \\
&\quad + a_2^k\big(\varphi_2^{k-1}a_2^{k-1} - \varphi_2^{k+1}a_2^{k+1} + \varphi_3^{k-2}a_3^{k-2} + \varphi_3^{k-1}a_3^{k-1} - \\
&\quad - \varphi_3^k a_3^k - \varphi_3^{k+1}a_3^{k+1}\big), \\
\dot{a}_3^k &= a_3^k\big(\varphi_1^k a_1^k - \varphi_1^{k+2}a_1^{k+2} + \varphi_2^{k-1}a_2^{k-1} + \varphi_2^k a_2^k - \varphi_2^{k+1}a_2^{k+1} - \varphi_2^{k+2}a_2^{k+2} + \\
&\quad + \varphi_3^{k-2}a_3^{k-2} + \varphi_3^{k-1}a_3^{k-1} - \varphi_3^{k+1}a_3^{k+1} - \varphi_3^{k+2}a_3^{k+2}\big).
\end{aligned} \tag{28}$$

Our goal has been achieved: we arrived at a bi-Hamiltonian manifold by means of a systematic procedure of reduction of the original Poisson bi-anchored manifold. Now we have to investigate the information provided by the bi-Hamiltonian structure.

## 4. The Toda$_3$ System

In this section we show how the bi-Hamiltonian structure obtained in the previous section defines specific vector fields and, moreover, accounts for their integrability. To obtain an integrable system we will focus on the Casimirs of the Poisson pencil, i.e., the functions $C$ such that their differentials are in the kernel of $P_\lambda$. Indeed, if we expand a Casimir in powers of $\lambda$,

$$C = \sum_i H_i \lambda^i, \tag{29}$$

it is immediate to check that the coefficients $H_i$ satisfy the Lenard relations

$$P'(\mathrm{d}H_i) = -P(\mathrm{d}H_{i+1}). \tag{30}$$

It is easily shown (see, e.g., [7]) that this in turn implies that the $H_i$'s are in involution with respect to both Poisson bracket. If these coefficients are enough, the

* We recall that a manifold endowed with two compatible Poisson tensors is said to be bi-Hamiltonian.

system is integrable in the classical sense of Liouville and Arnold [1]. Nevertheless, in general finding the Casimirs of a Poisson pencil is not easy. In the present case we can make use of the following

PROPOSITION 1. *If $h_k$ solves*

$$h_k h_{k+1} h_{k+2} = \left(a_1^{k+2} + \lambda\right) h_k h_{k+1} + a_2^{k+1} h_k + a_3^k, \tag{31}$$

*then $C(\lambda) = h_1 \cdots h_d$ is a Casimir of the Poisson pencil* (23)–(28). *The solutions $h_k$ and thus the Casimirs $C$ can be calculated explicitly as Laurent series in the parameter $\lambda$.*

*Proof.* See the appendix. □

We call (31) the *characteristic equation.* Proposition 1 allows us in principle to calculate the Casimirs of the Poisson pencil $\mathfrak{p}_\lambda$, but the computation is lengthy. Fortunately there is a shortcut: if in (31) we set

$$h_k = \frac{\psi_{k+1}}{\psi_k} \mu, \tag{32}$$

the characteristic equation becomes the linear system

$$0 = \psi_{k+3} \mu^3 - \left(a_1^{k+2} + \lambda\right) \psi_{k+2} \mu^2 - a_2^{k+1} \psi_{k+1} \mu - a_3^k \psi_k. \tag{33}$$

We can express (33) in matrix form as

$$L\psi = 0, \tag{34}$$

where $L$ is the matrix

$$L = \begin{pmatrix} \mu^2\left(a_1^1 + \lambda\right) & -\mu^3 & 0 & a_3^{d-1} & \mu a_2^d \\ \mu a_2^1 & \mu^2\left(a_1^2 + \lambda\right) & -\mu^3 & \ddots & a_3^d \\ a_3^1 & \mu a_2^2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \mu^2\left(a_1^{d-1} + \lambda\right) & -\mu^3 \\ -\mu^3 & 0 & a_3^{d-2} & \mu a_2^{d-1} & \mu^2\left(a_1^d + \lambda\right) \end{pmatrix}$$

and $\psi$ is the vector of the 'homogeneous coordinates'

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}.$$

For (34) to admit nontrivial solutions we must have $\det L = 0$. It can be proved that the cyclicity of the matrix $L$ implies that its determinant is a polynomial of degree 3 in $\mu^d$. Therefore we must have

$$0 = \det L = \pm\mu^{3d} + K_1 \mu^{2d} + K_2 \mu^d + K_3, \tag{35}$$

where $K_1$, $K_2$, $K_3$ are polynomials in $\lambda$ (in particular, $C_3$ does not depend on $\lambda$). By Proposition 1 and Equation (32), for all $\mu$ that satisfy (35) we have that $\mu^d = h_1 \ldots h_d$ is a Casimir, thus $K_1$, $K_2$ and $K_3$ are Casimirs as well. Their coefficients provide all information about the geometry of the system at hand.

The family of dynamical systems associated with the Casimirs $K_1$, $K_2$ and $K_3$ is what we call Toda$_3$. It can be proved that they coincide with Kupershmidt's reduction of the discrete KP hierarchy [5]. We will show in an example how the integrability of these systems stems from their bi-Hamiltonian structure.

## 5. An Example of the Toda$_3$ System

To illustrate how the scheme described above works we consider the specific case where $d = 4$. This example is easy to handle, but at the same time general enough. In order to make the equations easier to read we will change notation: we set $a_1^k = b_k$, $a_2^k = a_k$, $a_3^k = c_k$, $\varphi_1^k = \beta_k$, $\varphi_2^k = \alpha_k$, $\varphi_3^k = \gamma_k$. Thus our bi-Hamiltonian manifold $\mathcal{N}$ becomes $\mathbb{R}^{12}$ with coordinates $(a_1, \ldots, c_4)$. The Poisson pencil $\mathfrak{p}_\lambda$ associates with a one-form $\sum_{k=1}^4 (\alpha_k \mathrm{d}a_k + \beta_k \mathrm{d}b_k + \gamma_k \mathrm{d}c_k)$ the vector field

$$\begin{aligned}
\dot{b}_k &= (b_k + \lambda)(\alpha_{k-1}a_{k-1} - \alpha_k a_k + \gamma_{k+2}c_{k+2} - \gamma_k c_k) + \\
&\quad + a_k\beta_{k+1} - a_{k-1}\beta_{k-1} + c_k\alpha_{k+1} - c_{k+2}\alpha_{k+2}, \\
\dot{a}_k &= (c_{k-1}\alpha_{k-1} + a_k\beta_k)(b_k + \lambda) - \\
&\quad - (c_k\alpha_{k+1} + a_k\beta_{k+1})(b_{k+1} + \lambda) + c_k\beta_{k+2} - \beta_{k-1}c_{k-1} + \\
&\quad + a_k(\alpha_{k-1}a_{k-1} - \alpha_{k+1}a_{k+1} + \gamma_{k-1}c_{k-1} - \gamma_k c_k - \\
&\quad - \gamma_{k+1}c_{k+1} + \gamma_{k+2}c_{k+2}), \\
\dot{c}_k &= c_k(\beta_k(b_k + \lambda) - \beta_{k+2}(b_{k+2} + \lambda)) + \\
&\quad + c_k(-\alpha_{k+2}a_{k+2} + \alpha_k a_k - \alpha_{k+1}a_{k+1} + \alpha_{k-1}a_{k-1} + \\
&\quad + \gamma_{k-1}c_{k-1} - \gamma_{k+1}c_{k+1}),
\end{aligned} \tag{36}$$

where $k = 1, 2, 3, 4$. Equation (35) provides the Casimirs of the Poisson pencil (36). Indeed, from that equation one expects to find only three Casimirs of the whole pencil. The interesting feature of this approach is that in fact we obtain four Casimirs which prove to be enough to guarantee integrability. The three coefficients are

$$\begin{aligned}
K_1(\lambda) &= \lambda^4 + \lambda^3 C_1 + \lambda^2 H_1 + \lambda H_2 + H_3, \\
K_2(\lambda) &= \lambda^2 C_2 + \lambda C_3 + H_4, \\
K_3(\lambda) &= C_4,
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= b_1 + b_2 + b_3 + b_4, \\
H_1 &= a_1 + a_2 + a_3 + a_4 + b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1 + b_1 b_3 + b_2 b_4,
\end{aligned}$$

$$\begin{aligned}
H_2 &= c_1 + c_2 + c_3 + c_4 + b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_1 + b_4b_1b_2 + \\
&\quad + b_1(a_2 + a_3) + b_2(a_3 + a_4) + b_3(a_4 + a_1) + b_4(a_1 + a_2), \\
H_3 &= b_1b_2b_3b_4 + b_1c_2 + b_2c_3 + b_3c_4 + b_4c_1 + a_1a_3 + a_2a_4 + \\
&\quad + b_1b_2a_3 + b_2b_3a_4 + b_3b_4a_1 + b_4b_1a_2, \\
C_2 &= -c_2c_4 - c_1c_3, \\
C_3 &= -b_1c_2c_4 - b_2c_3c_1 - b_3c_4c_2 - b_4c_1c_3 + \\
&\quad + c_1a_3a_4 + c_2a_4a_1 + c_3a_1a_2 + c_4a_2a_3, \\
H_4 &= -a_1a_2a_3a_4 + b_1a_2a_3c_4 + b_2a_3a_4c_1 + b_3a_4a_1c_2 + b_4a_1a_2c_3 + \\
&\quad + a_1c_2c_3 + a_2c_3c_4 + a_3c_4c_1 + a_4c_1c_2 - b_2b_4c_1c_3 - b_1b_3c_2c_4, \\
C_4 &= c_1c_2c_3c_4.
\end{aligned}$$

Nonetheless, the second coefficient $K_2(\lambda)$ is composed itself of two independent Casimirs of the pencil: $\lambda^2 C_2$ and $\lambda C_3 + H_4$. Therefore we obtain the four Casimirs:

$$\begin{aligned}
K_1(\lambda) &= \lambda^4 + \lambda^3 C_1 + \lambda^2 H_1 + \lambda H_2 + H_3, \qquad K_2'(\lambda) = C_2, \\
K_2''(\lambda) &= \lambda C_3 + H_4, \qquad K_3(\lambda) = C_4.
\end{aligned}$$

In the theory of Gelfand–Zakharevich the important objects are these eight functions $C_1, \ldots, C_4$, $H_1, \ldots, H_4$. Since the $K$'s are Casimir functions of the Poisson pencil, we must have:

$$\begin{aligned}
\mathfrak{p}'(\mathrm{d}H_1) &= -\mathfrak{p}(\mathrm{d}C_1), \qquad \mathfrak{p}'(\mathrm{d}H_2) = -\mathfrak{p}(\mathrm{d}H_1), \\
\mathfrak{p}'(\mathrm{d}H_3) &= -\mathfrak{p}(\mathrm{d}H_2), \qquad \mathfrak{p}'(\mathrm{d}H_4) = -\mathfrak{p}(\mathrm{d}C_3)
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
0 &= \mathfrak{p}'(\mathrm{d}C_1) = \mathfrak{p}'(\mathrm{d}C_2) = \mathfrak{p}'(\mathrm{d}C_3) = \mathfrak{p}'(\mathrm{d}C_4) \\
&= \mathfrak{p}(\mathrm{d}C_2) = \mathfrak{p}(\mathrm{d}H_3) = \mathfrak{p}(\mathrm{d}C_4) = \mathfrak{p}(\mathrm{d}H_4).
\end{aligned} \tag{38}$$

The 4-particle periodic Toda$_3$ system are the vector fields $X_1, \ldots, X_4$ defined as follows:

$$X_1 = \mathfrak{p}'(\mathrm{d}H_1), \qquad X_2 = \mathfrak{p}'(\mathrm{d}H_2), \qquad X_3 = \mathfrak{p}'(\mathrm{d}H_3), \qquad X_4 = \mathfrak{p}'(\mathrm{d}H_4).$$

The explicit expression of these four vector fields is

$$\begin{aligned}
X_1\colon\quad & \dot{b}_1 = a_4 - a_1, \\
& \dot{a}_1 = a_1(b_2 - b_1) - c_1 + c_4, \\
& \dot{c}_1 = c_1(b_3 - b_1), \\
X_2\colon\quad & \dot{b}_1 = c_3 - c_1 - (b_3 + b_4)a_1 + (b_2 + b_3)a_4, \\
& \dot{a}_1 = a_1(-b_4b_1 - b_1b_3 + b_2b_3 + b_4b_2 - a_4 + a_2) - \\
& \qquad -c_1(b_4 + b_1) + c_4(b_2 + b_3), \\
& \dot{c}_1 = c_1(-b_4b_1 - b_1b_2 + b_2b_3 + b_3b_4 - a_4 - a_1 + a_2 + a_3), \\
X_3\colon\quad & \dot{b}_1 = -b_4c_1 + b_2c_3 - (a_3 + b_3b_4)a_1 + (a_2 + b_2b_3)a_4, \\
& \dot{a}_1 = a_1(-b_3b_4b_1 + b_2b_3b_4 - b_3a_4 - b_1a_3 + b_2a_3 + b_4a_2 -
\end{aligned}$$

$$
\begin{aligned}
&\quad -c_3 + c_2) - c_1(a_4 + b_4 b_1) + c_4(a_2 + b_2 b_3),\\
&\dot{c}_1 = c_1(-b_4 b_1 b_2 + b_2 b_3 b_4 - b_4 a_1 - b_2 a_4 + b_2 a_3 + b_4 a_2 - c_4 + c_2),\\
X_4\colon\quad &\dot{b}_1 = -c_1 c_4 a_3 + c_3 c_4 a_2,\\
&\dot{a}_1 = -c_4 b_1 c_1 a_3 - c_4 c_1 c_3 + c_1 b_2 c_4 a_3 + c_1 c_4 c_2,\\
&\dot{c}_1 = c_1(-a_4 a_1 c_2 + b_1 c_4 c_2 + a_2 a_3 c_4 - b_3 c_2 c_4).
\end{aligned}
$$

Of course, in the above formulas the cyclic condition holds and yields the other components. Due to (37) these vector fields are bi-Hamiltonian. In order to show that they are integrable we choose a symplectic leaf of the Poisson tensor $\mathfrak{p}'$. It can be shown that this leaf is given by

$$
\begin{aligned}
C_1 &= \text{constant}, & C_2 &= \text{constant},\\
C_3 &= \text{constant}, & C_4 &= \text{constant}.
\end{aligned}
$$

Therefore, the leaf is an eight-dimensional symplectic submanifold of the twelve-dimensional original phase space. As we said in Section 4, the functions $H_1, \dots, H_4$ commute with respect to both Poisson brackets. Therefore their restrictions to the symplectic leaf also commute, and, since they can be checked to be functionally independent, constitute an integrable system.

## 6. Conclusions

In this paper we showed that reducing a special Poisson bi-anchored manifold, namely the set of maps from $\mathbb{Z}_d$ to $\mathrm{GL}(3, \mathbb{R})$, we obtain a new bi-Hamiltonian structure. This bi-Hamiltonian structure gives rise to an integrable system, which we called Toda$_3$ and already appeared in the work of Kupershmidt [5]. This system represents a generalization of the periodic Toda lattice (which corresponds to $\mathrm{GL}(2, \mathbb{R})$).

There are several further developments of this approach. First of all, it is easy to endow the set of maps from $\mathbb{Z}_d$ to $\mathrm{GL}(n, \mathbb{R})$ for a generic $n \in \mathbb{N}$ with the structure of bi-anchored Poisson manifold. It is possible to show that the reduction of these manifolds gives rise to other Toda systems, which are the discrete analog of the Gelfand–Dickey hierarchies [3].

Secondly (see [8, 10]) the study of the conservation laws of the periodic Toda lattice allows to define the discrete analog [5] of the KP equations on the Sato Grassmannian [13]. These represent flows on an infinite-dimensional phase space that admit invariant submanifolds. These submanifolds are the different phase spaces of the Toda system, and the restriction of the KP equation to these phase spaces are the Toda equations. This way it is possible to extend to the discrete case the description given for the continuous case in [2], where the KdV hierarchy and the (usual) KP equations are considered.

**Appendix. Proof of Proposition 1**

We will split the proof in three parts.

(1) We are looking for Casimir functions, i.e. exact one-forms in the kernel of the Poisson pencil $\mathfrak{p}_\lambda$, which we rewrite like this

$$\begin{aligned}
\dot{a}_1^k &= \left(a_1^k+\lambda\right)\left(\varphi_2^{k-1}a_2^{k-1}-\varphi_2^k a_2^k+\varphi_3^{k-2}a_3^{k-2}-\varphi_3^k a_3^k\right)- \\
&\quad -\varphi_1^{k-1}a_2^{k-1}+\varphi_1^{k+1}a_2^k-\varphi_2^{k-2}a_3^{k-2}+\varphi_2^{k+1}a_3^k, \\
\dot{a}_2^k &= \varphi_2^{k-1}a_3^{k-1}\left(a_1^k+\lambda\right)-\varphi_2^{k+1}a_3^k\left(a_1^{k+1}+\lambda\right)- \\
&\quad -\varphi_1^{k-1}a_3^{k-1}+\varphi_1^{k+2}a_3^k+\varphi_3^{k-1}a_3^{k-1}a_2^k-\varphi_3^k a_3^k a_2^k+ \\
&\quad +a_2^k\left(\varphi_1^k\left(a_1^k+\lambda\right)-\varphi_1^{k+1}\left(a_1^{k+1}+\lambda\right)+\varphi_2^{k-1}a_2^{k-1}-\right. \\
&\quad \left.-\varphi_2^{k+1}a_2^{k+1}+\varphi_3^{k-2}a_3^{k-2}-\varphi_3^{k+1}a_3^{k+1}\right), \\
\dot{a}_3^k &= a_3^k\left(\varphi_1^k\left(a_1^k+\lambda\right)-\varphi_1^{k+1}\left(a_1^{k+1}+\lambda\right)+\varphi_2^{k-1}a_2^{k-1}-\right. \\
&\quad \left.-\varphi_2^{k+1}a_2^{k+1}+\varphi_3^{k-2}a_3^{k-2}-\varphi_3^{k+1}a_3^{k+1}\right)+ \\
&\quad +a_3^k\left(\varphi_1^{k+1}\left(a_1^{k+1}+\lambda\right)-\varphi_1^{k+2}\left(a_1^{k+2}+\lambda\right)+\varphi_2^k a_2^k-\right. \\
&\quad \left.-\varphi_2^{k+2}a_2^{k+2}+\varphi_3^{k-1}a_3^{k-1}-\varphi_3^{k+2}a_3^{k+2}\right).
\end{aligned} \tag{39}$$

From this expression, it is straightforward to see that in order for the one-form $\varphi$ to be in the kernel of the Poisson pencil it is enough that it satisfies the following equations:

$$\begin{aligned}
0 &= \left(a_1^k+\lambda\right)\left(\varphi_2^{k-1}a_2^{k-1}-\varphi_2^k a_2^k+\varphi_3^{k-2}a_3^{k-2}-\varphi_3^k a_3^k\right)- \\
&\quad -\varphi_1^{k-1}a_2^{k-1}+\varphi_1^{k+1}a_2^k-\varphi_2^{k-2}a_3^{k-2}+\varphi_2^{k+1}a_3^k, \\
0 &= \varphi_2^{k-1}a_3^{k-1}\left(a_1^k+\lambda\right)-\varphi_2^{k+1}a_3^k\left(a_1^{k+1}+\lambda\right)- \\
&\quad -\varphi_1^{k-1}a_3^{k-1}+\varphi_1^{k+2}a_3^k+\varphi_3^{k-1}a_3^{k-1}a_2^k-\varphi_3^k a_3^k a_2^k,
\end{aligned} \tag{40}$$

$$\begin{aligned}
0 &= \varphi_1^k\left(a_1^k+\lambda\right)-\varphi_1^{k+1}\left(a_1^{k+1}+\lambda\right)+\varphi_2^{k-1}a_2^{k-1}- \\
&\quad -\varphi_2^{k+1}a_2^{k+1}+\varphi_3^{k-2}a_3^{k-2}-\varphi_3^{k+1}a_3^{k+1}.
\end{aligned} \tag{41}$$

A change of variables reduces this system to a single equation of Riccati type. Indeed, let $h_k$ be any solution of what we call the *characteristic equation*, which is of Riccati type:

$$h_k h_{k+1} h_{k+2} = \left(a_1^{k+2}+\lambda\right)h_k h_{k+1} + a_2^{k+1}h_k + a_3^k. \tag{42}$$

Let us set

$$\varphi_1^{k+2} = h_k h_{k+1}\rho_k, \qquad \varphi_2^{k+1} = h_k\rho_k, \qquad \varphi_3^k = \rho_k.$$

Let us choose the multiplier $\rho$ is such a way that the following equation holds:

$$\varphi_1^k\left(a_1^k+\lambda\right)+\varphi_2^{k-1}a_2^{k-1}+\varphi_2^k a_2^k+\varphi_3^{k-2}a_3^{k-2}+\varphi_3^{k-1}a_3^{k-1}+\varphi_3^k a_3^k = L,$$

where $L$ is any constant different from zero.

(2) Such a one-form $\varphi$ solves (40) and therefore it is an element of the kernel of the Poisson pencil $\mathfrak{p}_\lambda$. Indeed, let us consider a one-form $\varphi$ that satisfies the system:

$$\begin{aligned}
&h_k h_{k+1} h_{k+2} = \left(a_1^{k+2} + \lambda\right) h_k h_{k+1} + a_2^{k+1} h_k + a_3^k,\\
&\varphi_1^k = h_{k-2} h_{k-1} \varphi_3^{k-2}, \qquad \varphi_2^k = h_{k-1} \varphi_3^{k-1},\\
&L = \varphi_1^k (a_1^k + \lambda) + \varphi_2^{k-1} a_2^{k-1} + \varphi_2^k a_2^k +\\
&\qquad + \varphi_3^{k-2} a_3^{k-2} + \varphi_3^{k-1} a_3^{k-1} + \varphi_3^k a_3^k.
\end{aligned} \tag{43}$$

We will show that $\varphi$ satisfies the following equations:

$$\begin{aligned}
0 \;=\;& \left(a_1^k + \lambda\right)\left(\varphi_2^{k-1} a_2^{k-1} - \varphi_2^k a_2^k + \varphi_3^{k-2} a_3^{k-2} - \varphi_3^k a_3^k\right) -\\
& - \varphi_1^{k-1} a_2^{k-1} + \varphi_1^{k+1} a_2^k - \varphi_2^{k-2} a_3^{k-2} + \varphi_2^{k+1} a_3^k,\\
0 \;=\;& \varphi_2^{k-1} a_3^{k-1}\left(a_1^k + \lambda\right) - \varphi_2^{k+1} a_3^k \left(a_1^{k+1} + \lambda\right) -\\
& - \varphi_1^{k-1} a_3^{k-1} + \varphi_1^{k+2} a_3^k + \varphi_3^{k-1} a_3^{k-1} a_2^k - \varphi_3^k a_3^k a_2^k,\\
0 \;=\;& \varphi_1^k \left(a_1^k + \lambda\right) - \varphi_1^{k+1}\left(a_1^{k+1} + \lambda\right) + \varphi_2^{k-1} a_2^{k-1} - \varphi_2^{k+1} a_2^{k+1} +\\
& + \varphi_3^{k-2} a_3^{k-2} - \varphi_3^{k+1} a_3^{k+1}.
\end{aligned} \tag{44}$$

We immediately see that $\varphi$ fulfills the third equation of (44): just compare with the last equation of (43). Before we verify that the two other equations are satisfied as well, we need a formula. Namely, we have ⋆

$$h_{k-3} h_{k-2} h_{k-1} \varphi_3^{k-3} - h_{k-2} h_{k-1}\left(a_1^k + \lambda\right)\varphi_3^{k-2} = h_{k-1} \varphi_3^{k-1} a_2^k + \varphi_3^k a_3^k. \tag{45}$$

We will show now that $\varphi$ satisfies the first equation of (44):

$$\begin{aligned}
0 \;=\;& \left(a_1^k + \lambda\right)\left(\varphi_2^{k-1} a_2^{k-1} - \varphi_2^k a_2^k + \varphi_3^{k-2} a_3^{k-2} - \varphi_3^k a_3^k\right) -\\
& - \varphi_1^{k-1} a_2^{k-1} + \varphi_1^{k+1} a_2^k - \varphi_2^{k-2} a_3^{k-2} + \varphi_2^{k+1} a_3^k.
\end{aligned} \tag{46}$$

---

⋆ From the last equation of (43) we obtain

$$\begin{aligned}
L \;=\;& \varphi_3^{k-2} h_{k-2} h_{k-1} (a_1^k + \lambda) + h_{k-2} \varphi_3^{k-2} a_2^{k-1} + h_{k-1} \varphi_3^{k-1} a_2^k + \varphi_3^{k-2} a_3^{k-2} +\\
& + \varphi_3^{k-1} a_3^{k-1} + \varphi_3^k a_3^k\\
=\;& \varphi_3^{k-2} (h_{k-2} h_{k-1} (a_1^k + \lambda) + h_{k-2} a_2^{k-1} + a_3^{k-2}) + h_{k-1} \varphi_3^{k-1} a_2^k + \varphi_3^{k-1} a_3^{k-1} + \varphi_3^k a_3^k\\
=\;& \varphi_3^{k-2} (h_{k-2} h_{k-1} h_k) + \varphi_3^{k-1} (h_{k-1} a_2^k + a_3^{k-1}) + \varphi_3^k a_3^k.
\end{aligned}$$

This implies

$$\begin{aligned}
&\varphi_3^{k-2} (h_{k-2} h_{k-1} h_k)\\
&\quad = \varphi_3^{k-1} (h_{k-1} h_k h_{k+1} - h_{k-1} a_2^k - a_3^{k-1}) + \varphi_3^k (h_k a_2^{k+1}) + \varphi_3^{k+1} a_3^{k+1}
\end{aligned}$$

and the result follows using the characteristic equation

$$h_{k-1} h_k h_{k+1} - a_2^k h_{k-1} - a_3^{k-1} = (a_1^{k+1} + \lambda) h_{k-1} h_k.$$

We replace in this expression the values of $\varphi_1^k$, $\varphi_2^k$ obtained from (43), and we multiply both sides by $h_{k-2}h_{k-1}$. We obtain that (46) holds if and only if

$$\begin{aligned} 0 \;=\; & h_{k-2}h_{k-1}\big(a_1^k+\lambda\big)\big(h_{k-2}\varphi_3^{k-2}a_2^{k-1} - h_{k-1}\varphi_3^{k-1}a_2^k + \varphi_3^{k-2}a_3^{k-2} - \varphi_3^k a_3^k\big) - \\ & - h_{k-2}h_{k-1}h_{k-3}h_{k-2}\varphi_3^{k-3}a_2^{k-1} + h_{k-2}h_{k-1}h_{k-1}h_k\varphi_3^{k-1}a_2^k - \\ & - h_{k-2}h_{k-1}h_{k-3}\varphi_3^{k-3}a_3^{k-2} + h_{k-2}h_{k-1}h_k\varphi_3^k a_3^k. \end{aligned}$$

Rearranging the terms in the last expression, we end up having to prove that

$$0 = \det\begin{pmatrix} \big(h_{k-3}h_{k-2}h_{k-1}\varphi_3^{k-3} - h_{k-2}h_{k-1}\big(a_1^k+\lambda\big)\varphi_3^{k-2}\big) & \big(h_{k-1}\varphi_3^{k-1}a_2^k + \varphi_3^k a_3^k\big) \\ \big(h_{k-2}h_{k-1}h_k - h_{k-2}h_{k-1}\big(a_1^k+\lambda\big)\big) & \big(h_{k-2}a_2^{k-1}a_3^{k-2}\big) \end{pmatrix}$$

but this is true, since the characteristic equation and (45) hold. We will show now that $\varphi$ satisfies the second equation of (44):

$$\begin{aligned} 0 \;=\; & \varphi_2^{k-1}a_3^{k-1}\big(a_1^k+\lambda\big) - \varphi_2^{k+1}a_3^k\big(a_1^{k+1}+\lambda\big) - \\ & - \varphi_1^{k-1}a_3^{k-1} + \varphi_1^{k+2}a_3^k + \varphi_3^{k-1}a_3^{k-1}a_2^k - \varphi_3^k a_3^k a_2^k. \end{aligned} \tag{47}$$

We replace in this expression the values of $\varphi_1^k$, $\varphi_2^k$ obtained from (43), and we multiply both sides by $h_{k-1}$. We obtain that (47) holds if and only if:

$$\begin{aligned} 0 \;=\; & h_{k-1}h_{k-2}\varphi_3^{k-2}a_3^{k-1}\big(a_1^k+\lambda\big) - h_{k-1}h_k\varphi_3^k a_3^k\big(a_1^{k+1}+\lambda\big) - \\ & - h_{k-1}h_{k-3}h_{k-2}\varphi_3^{k-3}a_3^{k-1} + h_{k-1}h_kh_{k+1}\varphi_3^k a_3^k + \\ & + h_{k-1}\varphi_3^{k-1}a_3^{k-1}a_2^k - h_{k-1}\varphi_3^k a_3^k a_2^k. \end{aligned}$$

Rearranging the terms in the last expression, we end up having to prove that

$$0 = \det\begin{pmatrix} a_3^{k-1} & \big(h_{k-1}h_kh_{k+1} - h_{k-1}h_k\big(a_1^{k+1}+\lambda\big) - a_2^k\big) \\ \varphi_3^k a_3^k & \big(h_{k-1}h_{k-3}h_{k-2}\varphi_3^{k-3} - h_{k-1}h_{k-2}\varphi_3^{k-2}\big(a_1^k+\lambda\big) - \varphi_3^{k-1}a_2^k\big) \end{pmatrix}$$

but this is true, since the characteristic equation and (45) hold.

(3) Furthermore, $\varphi$ is exact. The system

$$\begin{aligned} & h_kh_{k+1}h_{k+2} = \big(a_1^{k+2}+\lambda\big)h_kh_{k+1} + a_2^{k+1}h_k + a_3^k, \\ & \varphi_1^k = h_{k-2}h_{k-1}\varphi_3^{k-2}, \qquad \varphi_2^k = h_{k-1}\varphi_3^{k-1}, \\ & L = \varphi_1^k\big(a_1^k+\lambda\big) + \varphi_2^{k-1}a_2^{k-1} + \varphi_2^k a_2^k + \varphi_3^{k-2}a_3^{k-2} + \varphi_3^{k-1}a_3^{k-1} + \varphi_3^k a_3^k \end{aligned}$$

is equivalent to

$$\begin{aligned} & h_kh_{k+1}h_{k+2} = \big(a_1^{k+2}+\lambda\big)h_kh_{k+1} + a_2^{k+1}h_k + a_3^k, \\ & \varphi_1^k = h_{k-1}h_{k-2}\varphi_3^{k-2}, \qquad \varphi_2^k = h_{k-1}\varphi_3^{k-1}, \\ & 0 = L - \varphi_3^k a_3^k + \varphi_2^k\big(a_1^{k+1}+\lambda\big)h_k - \varphi_1^k h_k - \varphi_2^k h_kh_{k+1}. \end{aligned}$$

Combining the first and the fourth equations we obtain

$$\begin{aligned} & \varphi_3^k\big(a_1^{k+2}+\lambda\big)h_{k+1} + \varphi_3^k a_2^{k+1} \\ & \quad = -\frac{L}{h_k} + \varphi_3^k h_{k+1}h_{k+2} - \varphi_2^k(a_1^{k+1}+\lambda) + \varphi_1^k + \varphi_2^k h_{k+1}. \end{aligned}$$

We will use this in the next calculation. We move on to evaluating

$$
\begin{aligned}
&\sum_{k=1}^{d}\big(\varphi_1^{k+2}\dot a_1^{k+2}+\varphi_2^{k+1}\dot a_2^{k+1}+\varphi_3^k\dot a_3^k\big)\\
&\quad=\sum_{k=1}^{d}\big(h_kh_{k+1}\varphi_3^k\dot a_1^{k+2}+h_k\varphi_3^k\dot a_2^{k+1}+\varphi_3^k\dot a_3^k\big)\\
&\quad=\sum_{k=1}^{d}\varphi_3^k\big(h_kh_{k+1}\dot a_1^{k+2}+h_k\dot a_2^{k+1}+\dot a_3^k\big)\\
&\quad=\sum_{k=1}^{d}\varphi_3^k\begin{pmatrix}\dot h_kh_{k+1}h_{k+2}+h_k\dot h_{k+1}h_{k+2}+h_kh_{k+1}\dot h_{k+2}-\\ -\big(a_1^{k+2}+\lambda\big)\dot h_kh_{k+1}-\big(a_1^{k+2}+\lambda\big)h_k\dot h_{k+1}-a_2^{k+1}\dot h_k\end{pmatrix}\\
&\quad=\sum_{k=1}^{d}\begin{pmatrix}\varphi_3^k\dot h_kh_{k+1}h_{k+2}+\varphi_3^kh_k\dot h_{k+1}h_{k+2}+\varphi_3^kh_kh_{k+1}\dot h_{k+2}-\\ -\varphi_3^k\big(a_1^{k+2}+\lambda\big)\dot h_kh_{k+1}-\varphi_3^k\big(a_1^{k+2}+\lambda\big)h_k\dot h_{k+1}-\varphi_3^ka_2^{k+1}\dot h_k\end{pmatrix}\\
&\quad=\sum_{k=1}^{d}\begin{pmatrix}\varphi_3^k\dot h_kh_{k+1}h_{k+2}+\varphi_2^{k+1}\dot h_{k+1}h_{k+2}+\varphi_1^{k+2}\dot h_{k+2}-\\ -\dot h_k\big(\varphi_3^k(a_1^{k+2}+\lambda)h_{k+1}+\varphi_3^ka_2^{k+1}\big)-\varphi_2^{k+1}\big(a_1^{k+2}+\lambda\big)\dot h_{k+1}\end{pmatrix}\\
&\quad=\sum_{k=1}^{d}\begin{pmatrix}\varphi_3^k\dot h_kh_{k+1}h_{k+2}+\varphi_2^{k+1}\dot h_{k+1}h_{k+2}+\varphi_1^{k+2}\dot h_{k+2}-\\ -\dot h_k\Big(\varphi_1^k+\varphi_2^kh_{k+1}+\varphi_3^kh_{k+1}h_{k+2}-\dfrac{L}{h_k}-\varphi_2^k\big(a_1^{k+1}+\lambda\big)\Big)-\\ -\varphi_2^{k+1}\big(a_1^{k+2}+\lambda\big)\dot h_{k+1}\varphi_1^{k+2}\dot h_{k+2}.\end{pmatrix}\\
&\sum_{k=1}^{d}\begin{pmatrix}-\dot h_k\Big(\varphi_1^k+\varphi_2^kh_{k+1}+\varphi_3^kh_{k+1}h_{k+2}-\dfrac{L}{h_k}-\varphi_2^k\big(a_1^{k+1}+\lambda\big)-\\ -\varphi_3^kh_{k+1}h_{k+2}\Big)+\Big(\varphi_2^{k+1}h_{k+2}-\varphi_2^{k+1}\big(a_1^{k+2}+\lambda\big)\Big)\dot h_{k+1}\end{pmatrix}\\
&\quad=\sum_{k=1}^{d}\dot h_k\frac{L}{h_k}=\big(L\log(h_1\cdots h_d)\big)^{\bullet}.
\end{aligned}
$$

The last equation follows from $L$ being a constant (independent of both site $k$ and variables $a_i^k$). Therefore

$$
\sum_{k=1}^{d}\big(\varphi_1^k\mathrm{d}a_1^k+\varphi_2^k\mathrm{d}a_2^k+\varphi_3^k\mathrm{d}a_3^k\big)=\mathrm{d}(L\log(h_1\cdots h_d)).
$$

Thus, $\varphi$ is exact, and $C=h_1\dots h_d$ is a Casimir of the Poisson pencil. This ends the proof.

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