COMPUTATIONAL METHODS IN DECISION-MAKING, ECONOMICS AND FINANCE

COMPUTATIONAL METHODS IN DECISION-MAKING, ECONOMICS AND FINANCE
Optimization Models

Edited by

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Preface

Optimization is at the core of rational decision making. Even when the decision maker has more than one goal or there is significant uncertainty in the system, optimization provides a rational framework for efficient decisions. The Markowitz mean-variance framework is a classical example. This volume is devoted to recent developments in optimization decision models for finance and economics. The first four chapters focus directly on multi-stage problems in finance. Chapters 5-8 involve the use of worst-case analysis. Chapters 9-11 are devoted to portfolio optimization. The final four chapters are on transportation-inventory with stochastic demand; optimal investment with CRRA utility; hedging financial contracts; and, automatic differentiation for computational finance.

E.J. KONTOGHIORGHES, B. RUSTEM AND S. SIOKOS

This book is dedicated to our families.

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Chapter 1

MULTI-PERIOD OPTIMAL ASSET ALLOCATION FOR A MULTI-CURRENCY HEDGED PORTFOLIO*

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Abstract An asset allocation strategy is presented to support a fund manager who wants to outperform a constant weights, constant hedging benchmark. This strategy is a continuous time, multi-period extension of the classical one-period mean-variance optimization framework.

1. Introduction

The classical mean-variance approach to portfolio optimal asset allocation is intrinsically a one-period optimization: the investor sets a time horizon and maximizes his expected utility at that horizon. This can be good for a private investor, but certainly not for a fund manager whose goal is to outperform a benchmark. In this case there is no definite time horizon: the fund manager wants to "always" do better than the benchmark, so the best asset allocation for a fund manager is the outcome of an intertemporal optimization. The typical approach to multiperiod optimization relies on stochastic programming (see [3]). This approach is very flexible: setting a series of constraints one has

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control over a wide range of parameters, among which portfolio stability. The problem with stochastic programming is the heavy programming and machine time required, as well as the instability of the asset allocation as time elapses.

We develop a multiperiod model of portfolio allocation in continuous time. The fund manager aims at outperforming a constant weights, constant hedging benchmark by means of a stable combination of assets and hedging, assuming that he can rebalance continuously. The optimal asset allocation is then tested in a discrete time setting by means of Monte Carlo simulations. The paper is organized as follows. In Section 2 we derive some formulas for the evolution of a constant weights, constant hedging portfolios under the assumption that assets and foreign exchange rates evolve according to a lognormal stochastic process. We also determine the stochastic process followed by the ratio of a constant mix portfolio and a constant mix benchmark: this is the relative value that the fund manager wants to keep as large as possible. In Section 3 we define the utility function to maximize, and then we introduce the formalism that allows to calculate the optimal allocation. In Section 4 we illustrate in a practical example the outcome of the optimization. Finally, we discretize the rebalancing time step and show that the strategy's behavior is indistinguishable from the continuous time approximation. In Section 5 we conclude.

2. Portfolio dynamics

We start with a simple case: we consider a portfolio of n tradable assets, whose price per share we denote by $(X_1,\ldots,X_n)$. If we denote by $(\alpha_1,\ldots,\alpha_n)$ the number of shares of each asset, we can express the portfolio's value as

$$P(t) = \sum_{i=1}^{n} \alpha_i(t) X_i(t). \tag{1.1}$$

Our first aim is to determine the evolution of this portfolio under the following constraints:

- the relative weight of each asset in the portfolio is constant

$$\frac{\alpha_i(t) X_i(t)}{P(t)} = \omega_i; \tag{1.2}$$

- the portfolio is self-financing

An easy computation shows (see Appendix 1) that under the above assumptions the portfolio satisfies the following equation:

$$\frac{dP}{P} = \sum_{i=1}^{n} \omega_i \frac{dX_i}{X_i}. \tag{1.3}$$

This law is general: it does not depend on the process followed by asset prices. If we make the further assumption that the prices are lognormally distributed, i.e.,

$$\frac{dX_i}{X_i} = \mu_i dt + \sigma_i dW_i, \qquad dW_i dW_j = \rho_{ij} dt,$$

where $W_i(t)$ are standard Wiener processes in the objective measure, it is immediate to check that the portfolio is lognormally distributed, too

$$\frac{dP}{P} = \mu_P dt + \sigma_P dW, \tag{1.4}$$

where

$$\begin{aligned} \mu_P &= \sum_{i=1}^{n} \omega_i \mu_i \\ \sigma_P &= \sqrt{\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \omega_i \omega_j}. \end{aligned}$$

Now we are ready to analyze the kind of portfolio we are interested in, namely, a constant weights, constant hedging, multi-currency portfolio. Again, we consider a set of n tradable assets, whose price per share *in their local currency* we denote by $(X_1, \ldots, X_n)$. Again, we denote by $(\alpha_1, \ldots, \alpha_n)$ the number of shares of each asset respectively. Furthermore, we denote by Y_i the exchange rate with respect to the investor's currency. For example, if the investor is euro-based and asset X_2 is in yen, Y_2 is the number of euro necessary to buy a yen. With these notations, the value of the portfolio in the investor's currency reads

$$P(t) = \sum_{i=1}^{n} \alpha_i(t) X_i(t) Y_i(t). \tag{1.5}$$

In this case we want to determine the evolution of this portfolio under the following constraints:

- the relative weight of each asset i in the portfolio is constant

$$\frac{\alpha_i(t) X_i(t) Y_i(t)}{P(t)} = \omega_i; \tag{1.6}$$

- each asset i is hedged against exposure to its respective exchange rate risk. The hedging is a constant percentage h_i of the asset value;

- the portfolio is self-financing.

It is easy to prove (see Appendix 2) that under these assumptions the portfolio evolution is described by

$$\frac{dP}{P} = \sum_{i=1}^{n} \omega_i \left(\frac{dX_i}{X_i} + (1-h_i)\frac{dY_i}{Y_i} + \frac{dX_i}{X_i}\frac{dY_i}{Y_i} \right). \tag{1.7}$$

Again, in deriving (1.7) we did not make any assumption on the dynamics of assets and exchange rates. Assuming that they follow a lognormal process, i.e.[1],

$$\begin{aligned} \frac{dX_i}{X_i} &= \mu_i dt + \sigma_i dW_i \\ \frac{dY_i}{Y_i} &= \nu_i dt + \tau_i dZ_i \\ dW_i dW_j &= \rho_{ij} dt \\ dW_i dZ_j &= \xi_{ij} dt \\ dZ_i dZ_j &= \chi_{ij} dt, \end{aligned} \tag{1.8}$$

and applying Ito's rules, we obtain (see Appendix 2) that also the portfolio is lognormally distributed

$$\frac{dP}{P} = \mu_P dt + \sigma_P dW. \tag{1.9}$$

The explicit formula for drift and volatility of this lognormal process turn out to be

$$\mu_P = \omega' (\mu + H\nu + \mathbf{D}) \tag{1.10}$$

$$\sigma_P = \sqrt{\omega' \widetilde{H}' \Sigma \widetilde{H} \omega}, \tag{1.11}$$

where

$$\begin{aligned} \Sigma &= \begin{pmatrix} S & U \\ U' & T \end{pmatrix} \\ S_{ij} &= \sigma_i \sigma_j \rho_{ij}, U_{ij} = \sigma_i \tau_j \xi_{ij}, T_{ij} = \tau_i \tau_j \chi_{ij} \\ H &= \begin{pmatrix} 1-h_1 & 0 & \cdots & 0 \\ 0 & 1-h_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1-h_n \end{pmatrix} \\ \widetilde{H} &= \begin{pmatrix} I \\ H \end{pmatrix}, \mathbf{D} = \begin{pmatrix} U_{11} \\ \vdots \\ U_{nn} \end{pmatrix}. \end{aligned}$$

We derived so far the dynamics of a constant weights, constant hedging, multi-currency portfolio. In the next section we tackle the problem of finding the portfolio that ”best” outperforms a benchmark, which is also assumed to be a constant weights, constant hedging, multi-currency portfolio. Therefore it is useful to derive the evolution law of the ratio of two processes $P(t), B(t)$ (P will represent the portfolio and B the benchmark). Due to Ito’s lemma we have

$$\frac{d\left(\frac{P}{B}\right)}{\left(\frac{P}{B}\right)} = \frac{dP}{P} - \frac{dB}{B} - \frac{dP}{P}\frac{dB}{B} + \frac{dB}{B}\frac{dB}{B}. \tag{1.12}$$

This relation holds for generic $P(t), B(t)$. If these processes are lognormal, i.e.,

$$\begin{aligned} \frac{dP}{P} &= \mu_P dt + \sigma_P dW_P \\ \frac{dB}{B} &= \mu_B dt + \sigma_B dW_B \\ dW_P dW_B &= \rho_{PB} dt, \end{aligned} \tag{1.13}$$

a straightforward application of Ito’s rules yields (see Appendix 3) that the ratio of the two processes is also lognormal, and that its process is described by the following law

$$\frac{d\left(\frac{P}{B}\right)}{\left(\frac{P}{B}\right)} = \mu_{P/B} dt + \sigma_{P/B} dW, \tag{1.14}$$

where

$$\mu_{P/B} = \mu_P - \mu_B - \sigma_P \sigma_B \rho_{PB} + \sigma_B^2 \tag{1.15}$$

$$\sigma_{P/B} = \sqrt{\sigma_P^2 + \sigma_B^2 - 2\rho_{PB}\sigma_P\sigma_B}. \tag{1.16}$$

We are now in the position of computing the optimal asset allocation.

3. Optimal asset allocation

Suppose we want to form an optimal constant weights, constant hedging, multi-currency portfolio as in (1.5)

$$P(t) = \sum_{i=1}^{n} \alpha_i(t) X_i(t) Y_i(t), \tag{1.17}$$

where we denote by ω_i the constant weights and by h_i the constant hedging. Furthermore, suppose that the benchmark we are called to track and outperform is also a constant weights, constant hedging, multi-currency portfolio

$$B(t) = \sum_{i=1}^{n} \beta_i(t) X_i(t) Y_i(t), \tag{1.18}$$

where we denote by ϕ_i the constant weights and by k_i the constant hedging. In the lognormal economy described in the previous section, due to formulas (1.15) and (1.16) (1.10) (1.11), we know that the ratio P/B is lognormal. Using the above notation we can express the drift and volatility of this ratio as

$$\begin{aligned}
\mu_{P/B} &= \mu_P - \mu_B - \sigma_P\sigma_B\rho_{PB} + \sigma_B^2 \qquad (1.19)\\
&= \omega'(\mu + H\nu + \mathbf{D}) - \phi'(\mu + K\nu + \mathbf{D})\\
&- \omega'\widetilde{H}'\Sigma\widetilde{K}\phi + \phi'\widetilde{K}'\Sigma\widetilde{K}\phi\\
\sigma_{P/B}^2 &= \sigma_P^2 + \sigma_B^2 - 2\sigma_P\sigma_B\rho_{PB} \qquad (1.20)\\
&= \omega'\widetilde{H}'\Sigma\widetilde{H}\omega + \phi'\widetilde{K}'\Sigma\widetilde{K}\phi - 2\omega'\widetilde{H}'\Sigma\widetilde{K}\phi.
\end{aligned}$$

The drift $\mu_{P/B}$ represents the expected growth rate of portfolio P with respect to the benchmark B. The volatility $\sigma_{P/B}$ represents the tracking error of the portfolio with respect to the benchmark. In the light of a mean-variance approach to portfolio optimization we will seek the weights ω and hedges H that maximize the portfolio's growth rate minimizing the tracking error, according to a given value of risk aversion k. In formulas, we will solve

$$\max_{\omega,H}\left(\mu_{P/B} - k\sigma_{P/B}^2\right), \qquad (1.21)$$

where the dependence of the variables from the weights and hedges is expressed in (1.19) and (1.20).

We notice that problem (1.21) is somewhat similar to a more direct application of Markowitz's relative return optimization framework:

$$\max_{\omega,H}\left(\mu_{RR} - k\sigma_{RR}^2\right), \qquad (1.22)$$

where[2]

$$\begin{aligned}
\mu_{RR} &= \omega'(\mu + H\nu) - \phi'(\mu + K\nu) \qquad (1.23)\\
\sigma_{RR}^2 &= \omega'\widetilde{H}'\Sigma\widetilde{H}\omega + \phi'\widetilde{K}'\Sigma\widetilde{K}\phi - 2\omega'\widetilde{H}'\Sigma\widetilde{K}\phi. \qquad (1.24)
\end{aligned}$$

4. Empirical analysis

In this section we illustrate with a practical example the implementation of (1.21). We assume that the euro-based fund manager faces a constant weights, constant hedging benchmark of eleven assets exposed to three foreign currencies. Weights and exposure of the benchmark appear in the table below. The manager tries to outperform this benchmark by means of the constant mix asset allocation that solves (1.21), for a parameter of risk aversion $k = 5.25$, finding the set of optimal weights and hedges, also displayed in Table 1.1.

The first four assets represent global bond indices, the following six are global equity indices, then we have a liquidity index and the three currencies.

Table 1.1. Optimal portfolio allocation vs. benchmark

Assets/Fx	Potfolio	Benchmark
Area Euro	0.0%	30.8%
Area Pound	0.0%	1.7%
Area Yen	0.0%	7.1 %
Area Dollar	21.8%	10.4%
Europe ex-UK	11%	34.5%
UK	25.5%	2%
North America	22.4 %	10.6%
Asia ex-Japan	12.3%	0.6%
Japan	0.0%	2.4%
Emerging Markets	7.1%	0.0%
Cash	0.0%	0.0%
FX USD Hedging	84.3%	0.0%
FX GBP Hedging	90.4%	0.0%
FX JPY Hedging	0.0%	0.0%

Each asset is defined in local currency apart from equity Asia ex-Japan and Emerging Markets which are defined in US dollars.

We simulate a large number (20000) of Monte Carlo scenarios for the evolution of the price of the eleven assets and the value of the three exchange rates over a time span of a year, according to formula (1.8), where the input parameters are displayed in Table 1.2 (the correlation matrix is available upon request).

During the year, every determined time step, the benchmark rebalances, and so does the portfolio, sticking to the set of optimal weights and hedges displayed above. In the limit when rebalancing takes places continuously the above portfolio maximizes the utility function. When the rebalancing interval is discrete the empirical distribution of the outcomes of the simulations does not vary appreciably. This can be seen in Table 1.3.

Indeed, we performed the simulations with a rebalancing frequency of one day, one month and one quarter. To compare the outcomes, we displayed for these three sets of experiments the quantiles of the average over the year of the ratio of portfolio excess returns on the benchmark and tracking error (the so-called I Ratio) as well as the cumulative portfolio returns. The quantiles are

Table 1.2. Asset class simulation parameters

Assets/Fx	ExpReturns	Volatility
Area Euro	5.3%	3.6%
Area Pound	5.4%	6.2%
Area Yen	1.3%	4.1 %
Area Dollar	8.1%	4.3%
Europe ex-UK	11%	17.7%
UK	13.6%	17.1%
North America	13.6 %	19.6%
Asia ex-Japan	13%	18.5%
Japan	7.3%	20.2%
Emerging Markets	13%	20.0%
Cash	4.8%	0.1%
FX USD	0%	10.0%
FX GBP	0%	8.3%
FX JPY	0.0%	15%

Table 1.3. One-year distribution of key statistics in the case of discrete-time rebalancing

	Rebalancing: 1 day		Rebalancing: 1 month		Rebalancing: 1 quarter	
Quantile	IR	Cumul Return	IR	Cumul Return	IR	Cumul Return
20%	1.66	24%	1.65	24%	1.66	24%
30%	1.26	19%	1.25	19%	1.25	19%
40%	0.94	15%	0.94	15%	0.93	15%
50%	0.65	11%	0.65	11%	0.65	11%
60%	0.38	8%	0.37	8%	0.37	8%
70%	0.08	4%	0.08	4%	0.08	4%
80%	-0.25	0%	-0.25	0%	-0.25	0%
90%	-0.67	-5%	-0.69	-5%	-0.69	-5%

statistically indistinguishable, therefore the step from continuous (theoretical) to discrete (real-life) rebalancing does not affect the optimality of the strategy.

5. Conclusions

In this note we describe a multiperiod, continuous time theory of multi-currency, constant mix asset allocation. Then we test its application in discrete time. The theory relies on the assumption that asset prices and exchange rates follow lognormal stochastic process. This is a popular, nonetheless strong approximation to reality, that describes the whole evolution of the economy in terms of the first two moments of their joint distribution. Even so, drifts and covariance matrices are not known: they have to be estimated. Robustness analysis plays a key role in this framework, since it is well known [5] that optimization results are very sensitive to minimal changes in the input parameters. Furthermore, given the analytic expression for the evolution of constant mix portfolios (and benchmarks) (1.9) one might consider different utility functions: one option is the probability to beat the benchmark, as described by one of the authors [1]. Another interesting utility function stems from the similarity between the present theory and the direct application of a classical one-period mean-variance approach discussed above: as in [4], optimization in terms of relative returns on a benchmark implies suboptimal absolute returns; one might therefore consider a beta-constrained version of the problem. To conclude we hope that the computationally non intensive, stable asset allocation we propose in this paper will help some fund managers improve their performances.

Appendix

1. Constant weights, one-currency portfolios

The self-financing condition

$$\sum_{i=1}^{n} \alpha_i (X_i + dX_i) = \sum_{i=1}^{n} (\alpha_i + d\alpha_i)(X_i + dX_i) \tag{1.A.1}$$

simplifies to

$$\sum_{i=1}^{n} d\alpha_i (X_i + dX_i) = 0. \tag{1.A.2}$$

The evolution is described by

$$\begin{aligned} dP &= \sum_{i=1}^{n} d(\alpha_i X_i) = \sum_{i=1}^{n} d\alpha_i X_i + \alpha_i dX_i + d\alpha_i dX_i \\ &= \sum_{i=1}^{n} \alpha_i dX_i = \sum_{i=1}^{n} \alpha_i X_i \frac{dX_i}{X_i}, \end{aligned}$$

where we made use of equation $(1.A.2)$. The last expression implies

$$\frac{dP}{P} = \sum_{i=1}^{n} \frac{\alpha_i X_i}{P} \frac{dX_i}{X_i} = \sum_{i=1}^{n} \omega_i \frac{dX_i}{X_i}. \tag{1.A.3}$$

This is the evolution law of the portfolio, no matter what the process for the asset prices is. We will now make the further assumption that the assets are lognormally distributed. This means

$$\frac{dX_i}{X_i} = \mu_i dt + \sigma_i dW_i, \qquad dW_i dW_j = \rho_{ij} dt.$$

Under this condition it is immediate to check that the portfolio is lognormally distributed

$$\frac{dP}{P} = \mu_P dt + \sigma_P dW, \tag{1.A.4}$$

where

$$\begin{aligned} \mu_P &= \sum_{i=1}^{n} \omega_i \mu_i \\ \sigma_P &= \sqrt{\sum_{i,j=1}^{n} \omega_i \sigma_i \sigma_j \rho_{ij} \omega_j}. \end{aligned}$$

2. Constant weights, constant hedging, multi-currency portfolios

We will first write the conditions self-financing and constant hedge in analytical terms. Consider the portfolio (1.5) at a generic time t and suppose a percentage h_i of each asset i is hedged against exposure to the currency exchange rate risk. This means that in the infinitesimal time period dt the portfolio value evolves as follows:

$$P = \sum_{i=1}^{n} \alpha_i X_i Y_i \longmapsto \sum_{i=1}^{n} \left(\alpha_i X_i Y_i + \alpha_i d\left(X_i Y_i\right) - h_i \alpha_i X_i dY_i\right).$$

At the end of the infinitesimal time period dt the investor rebalances her portfolio, buying shares and new hedging. The cost of hedging is zero, though. Therefore, the self-financing condition reads

$$\begin{aligned} &\sum_{i=1}^{n} \left(\alpha_i X_i Y_i + \alpha_i d\left(X_i Y_i\right) - h_i \alpha_i X_i dY_i\right) = \\ &= \sum_{i=1}^{n} \left(\alpha_i + d\alpha_i\right)\left(X_i + dX_i\right)\left(Y_i + dY_i\right) \end{aligned} \tag{1.A.5}$$

and the condition of constant hedging simply means $h_i = \text{constant}_i$. We can simplify $(1.A.5)$. The l.h.s. reads

$$\sum_{i=1}^{n} \left(\alpha_i X_i Y_i + \alpha_i dX_i Y_i + \alpha_i X_i dY_i + \alpha_i dX_i dY_i - h_i \alpha_i X_i dY_i\right)$$

and the r.h.s. is

$$\begin{aligned} & \sum_{i=1}^{n} \left(\alpha_i \left(X_i + dX_i\right)\left(Y_i + dY_i\right) + d\alpha_i \left(X_i + dX_i\right)\left(Y_i + dY_i\right)\right) = \\ = & \sum_{i=1}^{n} \left(\alpha_i X_i Y_i + \alpha_i dX_i Y_i + \alpha_i X_i dY_i + \alpha_i dX_i dY_i + \right. \\ + & \left. d\alpha_i \left(X_i + dX_i\right)\left(Y_i + dY_i\right)\right). \end{aligned} \tag{1.A.6}$$

Therefore $(1.A.5)$ simplifies to:

$$\sum_{i=1}^{n} h_i \alpha_i X_i dY_i = -\sum_{i=1}^{n} d\alpha_i X_i Y_i + d\alpha_i dX_i Y_i + d\alpha_i X_i dY_i + d\alpha_i dX_i dY_i. \tag{1.A.7}$$

We can use these constraints to determine the portfolio evolution:

$$\begin{aligned} dP &= \sum_{i=1}^{n} d\left(\alpha_i X_i Y_i\right) \\ &= \sum_{i=1}^{n} \left(\alpha_i dX_i Y_i + \alpha_i X_i dY_i + \alpha_i dX_i dY_i + d\alpha_i X_i Y_i\right. \\ &+ \left. d\alpha_i dX_i Y_i + d\alpha_i X_i dY_i + d\alpha_i dX_i dY_i \right) \\ &= \sum_{i=1}^{n} \left(\alpha_i dX_i Y_i + (1-h_i)\,\alpha_i X_i dY_i + \alpha_i dX_i dY_i\right), \end{aligned}$$

where we made use of equation $(1.A.7)$. The last expression implies

$$\begin{aligned} \frac{dP}{P} &= \sum_{i=1}^{n} \frac{\alpha_i X_i Y_i}{P}\frac{dX_i}{X_i} + (1-h_i)\,\frac{\alpha_i X_i Y_i}{P}\frac{dY_i}{Y_i} + \frac{\alpha_i X_i Y_i}{P}\frac{dX_i}{X_i}\frac{dY_i}{Y_i} \\ &= \sum_{i=1}^{n} \omega_i \left(\frac{dX_i}{X_i} + (1-h_i)\,\frac{dY_i}{Y_i} + \frac{dX_i}{X_i}\frac{dY_i}{Y_i}\right), \end{aligned} \tag{1.A.8}$$

where we made use of (1.6). Now we assume lognormality (1.8): substituting this condition in $(1.A.7)$ and applying Ito's multiplication rules we get

$$\begin{aligned} \frac{dP}{P} &= \sum_{i=1}^{n} \omega_i\left(\mu_i dt + \sigma_i dW_i\right) + (1-h_i)\,\omega_i\left(\nu_i dt + \tau_i dZ_i\right) + \omega_i\left(\mu_i dt + \sigma_i dW_i\right)\left(\nu_i dt + \tau_i dZ_i\right) \\ &= \sum_{i=1}^{n} \omega_i \mu_i dt + (1-h_i)\,\omega_i \nu_i dt + \omega_i \sigma_i dW_i \tau_i dZ_i + \omega_i \sigma_i dW_i + (1-h_i)\,\omega_i \tau_i dZ_i \\ &= \sum_{i=1}^{n} \left(\omega_i \mu_i + (1-h_i)\,\omega_i \nu_i + \omega_i \sigma_i \tau_i \xi_{ii}\right) dt + \omega_i \sigma_i dW_i + (1-h_i)\,\omega_i \tau_i dZ_i. \end{aligned}$$

Therefore

$$\frac{dP}{P} = \mu_P dt + \sigma_P dW,$$

where

$$\begin{aligned} \mu_P &= \sum_{i=1}^{n} \left(\omega_i \mu_i + (1-h_i)\,\omega_i \nu_i + \omega_i \sigma_i \tau_i \xi_{ii}\right) \\ \sigma_P^2 &= \sum_{i,j=1}^{n} \left(\omega_i \omega_j \sigma_i \sigma_j \rho_{i_j} + \omega_i \sigma_i \left(1-h_j\right) \omega_j \tau_j \xi_{ij} + (1-h_i)\left(1-h_j\right) \omega_i \omega_j \tau_i \tau_j \chi_{ij}\right). \end{aligned}$$

3. The evolution of the ratio of two lognormal processes

By simple substitution of (1.13) into (1.12) we obtain

$$\begin{aligned}
\frac{d\left(\frac{P}{B}\right)}{\left(\frac{P}{B}\right)} &= \frac{dP}{P} - \frac{dB}{B} - \frac{dP}{P}\frac{dB}{B} + \frac{dB}{B}\frac{dB}{B} \\
&= \mu_P dt + \sigma_P dW_P - \mu_B dt - \sigma_B dW_B \\
&- (\mu_P dt + \sigma_P dW_P)(\mu_B dt + \sigma_B dW_B) \\
&+ (\mu_B dt + \sigma_B dW_B)(\mu_B dt + \sigma_B dW_B) \\
&= \mu_P dt + \sigma_P dW_P - \mu_B dt - \sigma_B dW_B - \sigma_P \sigma_B \rho_{PB} dt + \sigma_B^2 dt \\
&= \left(\mu_P - \mu_B - \sigma_P \sigma_B \rho_{PB} + \sigma_B^2\right) dt + \sigma_P dW_P - \sigma_B dW_B.
\end{aligned}$$

This proves our statement.

Notes

1. This notation includes the case where asset i's currency and asset j's currency are the same, in which case

$$\nu_i = \nu_j, \qquad \tau_i = \tau_j, \qquad \chi_{ij} = 1.$$

2. This is the usual framework as found, e.g. in Hull [2]. A more precise expression should take the second order effect of the covariance of FX and asset returns into consideration.

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